Relative Bott-Chern Secondary Characteristic Classes

Lin Weng

Department of Mathematics, Graduate School of Science, Osaka University,
Toyonaka, Osaka 560, Japan

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Abstract. In this paper, we introduce six axioms for relative Bott-Chern secondary characteristic classes and prove the uniqueness and existence theorem for them. Such a work provides us a natural way to understand and hence to prove the arithmetic Grothendieck-Riemann-Roch theorem.

Content

Introduction

This paper comes partly as a result of our renewed attemp to simplify and clarify our previous work on the so-called relative Bott-Chern secondary characteristic classes and an arithmetic Grothendieck-Riemann-Roch theorem for local complete intersection morpohisms done in 1991 (see e.g., [We1,2]). Our (initial and final) aim is to expose the natural structure of the arithmetic Grothendieck-Riemann-Roch theorem via a theory for relative Bott-Chern secondary characteristic classes.

Yet, the works done in [We1,2] are far from being perfect. To make things worse, as what we found recently, the original uniqueness assertion for the relative Bott-Chern secondary characteristic classes stated in [We1,2] is not entirely correct. Fortunately, this mistake is by no means a fatal one. It is our main task here to correct this mistake and to fix some other flaws (appeared in [We1,2]) so as to make the theory of relative Bott-Chern secondary characteristic classes and our arithmetic Grothendieck-Riemann-Roch theorem for local complete intersection morphisms revive. In this way, we then hope that the reader can understand the arithmetic Grothendieck-Riemann-Roch theorem at its own disposal.

This paper is organized as follows. We begin with a review of a theory of the classical Bott-Chern secondary characteristic classes in Chapter 1. Then, in Chapter 2, we introduce five of six key axioms for our relative Bott-Chern secondary characteristic classes, which consist of the downstairs rule, the projection rule, the functorial rule, the uniqueness rule with respect to hermitian vector bundles, and the uniqueness rule with respect to metrized morphisms. The final axiom, the deformation to the normal cone rule, for relative Bott-Chern secondary characteristic classes is introduced in Chapter 3. After this, we state two uniqueness theorems for the relative Bott-Chern secondary characteristic classes in Chapter 4. In Chapter 5, we prove a few intermediate results as a preparation to prove the uniqueness theorems. In Chapter 6, we complete the proof of the uniqueness theorems for relative Bott-Chern secondary characteristic classes. Finally, in Chapter 7, we prove a weak version of the existence theorem for relative Bott-Chern secondary characteristic classes by an effective construction, which is sufficient and necessary for the application to the arithmetic Grothendieck-Riemann-Roch theorem.

1. Classical Bott-Chern secondary characteristic classes

A) Axioms for classical Bott-Chern classes

B) Uniqueness of classical Bott-Chern classes

C) Existence of classical Bott-Chern classes

Associated to a hermitian vector bundle (E, ρ) on a complex manifold M are Chern forms $\mathrm{ch}(E, \rho)$. It is well-known that the differential forms $\mathrm{ch}(E, \rho)$ are d-closed on M. Hence they define de Rham cohomology classes $[\mathrm{ch}(E, \rho)] \in H^*(M, \mathbb{R})$. Moreover, the classes $[\mathrm{ch}(E, \rho)]$ do not depend on the choice of the metric ρ.

On the other hand, if we view $\mathrm{ch}(E, \rho)$ as genuine differential forms, they do depend on the metric ρ. Hence a natural question is to understand how Chern forms depend on metrics. This problem is solved by Bott and Chern in [BC], via the so-called classical Bott-Chern secondary characteristic classes. We in this chapter review the accociated theory.

A. Axioms for classical Bott-Chern classes

(A.1) Let (E, ρ) be a hermitian vector bundle of rank r on a complex manifold M. It is well-known that there exists a unique connection $\nabla_{E,\rho}$ on E which is compatible with the complex structure $\bar{\partial}$ of E and preserves the metric ρ. The associated curvature is then defined to be $\nabla^2_{E,\rho}$.

With respect to a local frame s_U over an open trivialization chart U for E, the connection $\nabla_{E,\rho}$ (resp. the curvature $\nabla^2_{E,\rho}$) may be represented by an $r \times r$-matrix ω_U (resp. Ω_U) of differential 1-forms (resp. differential (1,1)-forms) on U. Furthermore, Ω_U satisfies the *Bianchi identity*

$$d\Omega = \Omega \wedge \omega - \omega \wedge \omega. \tag{1.1}$$

Moreover, if V is another open trivialization chart for E with s_V a local frame of E on V, $g_{VU} : U \cap V \to \mathrm{GL}(r, \mathbb{C})$, the associated holomorphic transformation, then on $U \cap V$, $s_U = s_V \cdot g_{VU}$, and

$$\Omega_U = g_{VU}^{-1} \cdot \Omega_V \cdot g_{VU}. \tag{1.2}$$

Here Ω_V denotes the curvature forms of (E, ρ) on V with respect to the local frame s_V.

(A.2) To facilitate the ensuring discussion, we now recall a result from algebra. Let $B \subset \mathbb{R}$ be a subring, and let $\phi \in B[[T_1, \ldots, T_r]]$ be a symmetric power series. For

every $k \geq 0$, denote by $\phi_{[k]}$ the degree k homogeneous component of ϕ. Then there exists a unique polynomial map

$$\Phi_{[k]} : M_r(\mathbb{C}) \to \mathbb{C}$$

from the ring of $r \times r$ complex matrices such that

(1) $\Phi_{[k]}$ is invariant under the conjugation of $\mathrm{GL}(n, \mathbb{C})$;
(2) $\Phi_{[k]}(\mathrm{diag}(a_1, \ldots, a_r)) = \phi_{[k]}(a_1, \ldots, a_r)$.

Thus, more generally, for any B-algebra A, we have a well-defined map

$$\Phi = \oplus_{k\geq 0}\Phi_{[k]} : M_r(A) \to A.$$

This then implies that for a nilpotent subalgebra I of A, we may also have a well-defined map

$$\Phi = \oplus_{k\geq 0}\Phi_{[k]} : M_r(I) \to A.$$

(A.3) We apply (A.1) and (A.2) as follows. With the same notation as above, if (E, ρ) is a hermitian vector bundle of rank r on a complex manifold M, and $\phi \in B[[T_1, \ldots, T_r]]$ is a symmetric power series, then on a local trivialization chart U of E, define differential forms $\phi(E, \rho; U)$ on U by

$$\phi(E, \rho; U) := \Phi(\frac{1}{2\pi\sqrt{-1}} \cdot \Omega_U).$$

Here we set $A := A(U) := \oplus_{p\geq 0} A^{p,p}(U)$ be the space of all (p, p)-differential forms on U and $I =: \oplus_{p\geq 1} A^{p,p}(U)$.

Then by (1.2), and A.2.1, $\{(U, \phi(E, \rho; U))\}$ defines global differential forms on M. Denote these resulting forms by $\phi(E, \rho)$ and call them the *characteristic forms* assocated to (E, ρ) with respect to ϕ.

Theorem. *With the same notation as above,*

(1) $\phi(E, \rho)$ are d-closed differential forms on M, i.e. $d\big(\phi(E, \rho)\big) = 0$.

(2) For any map $f : N \to M$ of complex manifolds, $f^*\big(\phi(E, \rho)\big) = \phi(f^*E, f^*\rho)$.

(3) If (E, ρ) is a hermitian line bundle (L, ρ), then $\phi(L, \rho) = \phi(c_1(L, \rho))$. Here $c_1(L, \rho)$ is the so-called first Chern form of (L, ρ) defined to be the differential form $dd^c\big(-\log \|s\|_\rho^2\big)$.

(4) The de Rham cohomology classes $[\phi(E, \rho)] \in H^*(M, \mathbb{R})$ of $\phi(E, \rho)$ do not depend on the choice of ρ, but the forms $\phi(E, \rho)$ themselves do depend on ρ.

The proof of this theorem may be found in standard textbooks which contain the theory of characteristic forms. (In fact, (1) is a direct consequence of the Bianchi identity (1.1).) So we omit it. Instead, we point out that first, (1), (2), (3) and (4) charactrize the characteristic classes associated to ϕ uniquely; and secondly, (4) is the starting point of our story.

(A.4) Now a natural question is how characteristic forms depend on hermitian metrics. This problem is solved by Bott and Chern via the so-called the classical Bott-Chern secondary characteristic classes [BC].

To state their fundamental result, following [GS2], we start with the following axioms for the classical Bott-Chern secondary characteristic classes, $\phi_{\mathrm{BC}}(E.,\rho.)$, with values in $\tilde{A}(M) := A(M)/(\mathrm{Im}\partial + \mathrm{Im}\bar{\partial})$, associated to a symmetric power series ϕ, a short exact sequence of vector bundles $E. : 0 \to E_1 \to E_2 \to E_3 \to 0$ on M, and hermitian metrics ρ_j on E_j for $j = 1, 2, 3$.

Axiom 1. (**Downstairs Rule**) In $A(M)$,

$$dd^c\phi_{\mathrm{BC}}(E.,\rho.) = \phi(E_2,\rho_2) \; - \; \phi(E_1 \oplus E_3, \rho_1 \oplus \rho_3).$$

Axiom 2. (**Functorial Rule**) For any morphism $f : N \to M$ of complex manifolds,

$$f^*\big(\phi_{\mathrm{BC}}(E.,\rho.)\big) \; = \; \phi_{\mathrm{BC}}(f^*E., f^*\rho.).$$

Axiom 3. (**Uniqueness Rule**) If $(E.,\rho.)$ splits, i.e. $(E_2,\rho_2) = (E_1 \oplus E_3, \rho_1 \oplus \rho_3)$, then

$$\phi_{\mathrm{BC}}(E.,\rho.) = 0.$$

(A.5) We have the following

Theorem. (**Existence and Uniqueness of Classical Bott-Chern Secondary Characteristic Classes**) ([BC] & [GS2]) *For any symmetric power series ϕ, there exists a unique construction ϕ_{BC} such that associated to each short exact sequence of vector bundles on a complex manifold M*

$$E. : \qquad 0 \to E_1 \to E_2 \to E_3 \to 0$$

and hermitian metrics ρ_j on E_j for $j = 1, 2, 3$, is a unique element $\phi_{\mathrm{BC}}(E.,\rho.) \in \tilde{A}(M)$ which satisfies Axioms 1, 2, and 3.

Remark 1.1. We will call the unique element $\phi_{\mathrm{BC}}(E.,\rho.) \in \tilde{A}(M)$ the *classical Bott-Chern secondary characteristic classes*, or *classical Bott-Chern classes* or simply

Bott-Chern classes, associated to $(E., \rho.)$ and ϕ. Useful examples are given when ϕ corresponds to ch, the Chern characteristic class, or td, the Todd characteristic class. In these cases, we denote them by $\mathrm{ch}_{\mathrm{BC}}$ and $\mathrm{td}_{\mathrm{BC}}$ respectively.

Remark 1.2. $\phi_{\mathrm{BC}}(E., \rho.)$'s are not differential forms, rather they are in $\tilde{A}$, the quotient space of differential forms modulo the exact forms.

Remark 1.3. When $E_3 = 0$, so that $E_1 = E_2 =: E$, we write $\phi_{\mathrm{BC}}(E., \rho.)$ simply as $\phi_{\mathrm{BC}}(E; \rho_2, \rho_1)$. Thus, by Axiom 1,

$$dd^c \phi_{\mathrm{BC}}(E; \rho_2, \rho_1) = \phi(E, \rho_2) - \phi(E, \rho_1).$$

So $\phi_{\mathrm{BC}}(E; \rho_2, \rho_1)$ measures how $\phi(E, \rho)$ depends on the metric ρ.

B. Uniqueness of classical Bott-Chern classes

(B.1) For the time being, assume that there exist elements $\phi_{\mathrm{BC}}(E., \rho.) \in \tilde{A}(M)$ satisfying Axioms 1, 2 and 3. Our aim in this section is to prove the uniqueness part of Theorem A.5 by using a $\mathbb{P}^1$-deformation following [GS2].

Let 1_∞ be a section of the bundle $\mathcal{O}_{\mathbb{P}^1}(1)$, which vanishes at ∞ and has the value 1 at 0. So we have an inclusion $\mathrm{Id}_{E_1} \otimes 1_\infty : E_1 \to E_1(1)$. Set

$$DE_1 := E_1(1) := E_1 \otimes \mathcal{O}_{\mathbb{P}^1}(1), \quad DE_2 := (E_2 \oplus E_1(1))/E_1, \quad DE_3 := E_3.$$

Then we obtain an exact sequence of vector bundles on $M \times \mathbb{P}^1$:

$$DE. : \qquad 0 \to DE_1 \to DE_2 \to DE_3 \to 0.$$

Easily we see that the restriction of $DE.$ on $M \times \{0\}$ gives $E.$ on M while the restrction of $DE.$ on $M \times \{\infty\}$ gives the splitting exact sequence $0 \to E_1 \to E_1 \oplus E_3 \to E_3 \to 0$ on M. (So we deform $E.$ to a split exact sequence.) By a partition of unity, we may further choose hermitian metrics $D\rho_i$ on DE_i, $i = 1, 2, 3$, such that the induced metrics on $M \times \{0\}$ (resp. on $M \times \{\infty\}$) via restrictions coincide with the original metrics ρ_i (resp. splits). Now define an element in $A(M)$ by setting

$$\phi'_{\mathrm{BC}}(E., \rho.) := \int_{\mathbb{P}^1} [\log |z|^2] \cdot \Big(\phi(DE_2, D\rho_2) \Big).$$

By an abuse of notation, we use $\phi'_{\mathrm{BC}}(E., \rho.)$ to denote its image in $\tilde{A}(M)$ as well.

To prove the uniqueness, it suffices to justify the following

Claim. *With the same notation as above,*

(1) $\phi'_{\mathrm{BC}}(E.,\rho.) \in \tilde{A}(M)$ *does not depend on* $D\rho_2$*;*

(2) $\phi'_{\mathrm{BC}}(E.,\rho.) = \phi_{\mathrm{BC}}(E.,\rho.)$ *in* $\tilde{A}(M)$*.*

(B.2) To prove Claim B.1.1, we use a $\mathbb{P}^1$-deformation again. Suppose that $D\rho_2$ and $D\rho'_2$ are metrics on DE_2 such that they induce the same metrics on $M \times \{0\}$ as well as on $M \times \{\infty\}$. Consider the product $M \times \mathbb{P}^1 \times \mathbb{P}^1$ with points (y,z,u). We have the following natural maps:

$$M \times \mathbb{P}^1 \overset{i_u^{12}}{\hookrightarrow} M \times \mathbb{P}^1 \times \mathbb{P}^1 \overset{p_{12}}{\rightarrow} M \times \mathbb{P}^1, \quad M \times \mathbb{P}^1 \overset{i_z^{13}}{\hookrightarrow} M \times \mathbb{P}^1 \times \mathbb{P}^1 \overset{p_{13}}{\rightarrow} M \times \mathbb{P}^1,$$

defined by

$$i_u^{12}(y,z) := (y,z,u),\ \ p_{12}(y,z,u) := (y,z), \quad i_z^{13}(y,u) := (y,z,u),\ \ p_{13}(y,z,u) := (y,u).$$

Also let $p_1 : M \times \mathbb{P}^1 \to M$ be the projection to the first factor. Then by a partition of unity, we may find a metric τ on the bundle $p_{12}^* DE_2$ such that

(i) $(i_0^{12})^*(p_{12}^* DE_2, \tau) \simeq (DE_2, D\rho_2)$ and $(i_\infty^{12})^*(p_{12}^* DE_2, \tau) \simeq (DE_2, D\rho'_2)$;

(ii) $(i_0^{13})^*(p_{12}^* DE_2, \tau) \simeq p_1^*(E_2, \rho_2)$ and $(i_\infty^{13})^*(p_{12}^* DE_2, \tau) \simeq p_1^*(E_1 \oplus E_3, \rho_1 \oplus \rho_3)$.

Hence,

$$\begin{aligned}
&\int_{\mathbb{P}^1} [\log|z|^2]\, \phi(DE_2, D\rho_2) - \int_{\mathbb{P}^1} [\log|z|^2]\, \phi(DE_2, D\rho'_2) \\
=& \int_{\mathbb{P}^1} [\log|z|^2] \Big(\phi(DE_2, D\rho_2) - \phi(DE_2, D\rho'_2) \Big) \\
=& \int_{\mathbb{P}^1} [\log|z|^2] \Big(\phi\big((i_0^{12})^*(p_{12}^* DE_2, \tau)\big) - \phi\big((i_\infty^{12})^*(p_{12}^* DE_2, \tau)\big) \Big) \\
=& \int_{\mathbb{P}^1 \times \mathbb{P}^1} [\log|z|^2]\, [\log|u|^2] \Big(d_u d_u^c \big(\phi(p_{12}^* DE_2, \tau) \big) \Big) \quad \text{(by Stokes' formula)}.
\end{aligned}$$

But if we let $\partial = \partial_M + \partial_z + \partial_u$ and $\bar\partial = \bar\partial_M + \bar\partial_z + \bar\partial_u$ be the differentials on $M \times \mathbb{P}^1 \times \mathbb{P}^1$, then by the fact that characteristic forms are d-closed, we have

$$\begin{aligned}
&\int_{\mathbb{P}^1 \times \mathbb{P}^1} [\log|z|^2]\, [\log|u|^2] \Big(d_u d_u^c \big(\phi(p_{12}^* DE_2, \tau) \big) \Big) \\
&\qquad = \int_{\mathbb{P}^1 \times \mathbb{P}^1} [\log|z|^2]\, [\log|u|^2] \Big(d_z d_z^c \big(\phi(p_{12}^* DE_2, \tau) \big) \Big).
\end{aligned}$$

Thus, using Stokes' formula again, we have

$$
\begin{aligned}
&\int_{\mathbb{P}^1} [\log|z|^2]\, \phi(DE_2, D\rho_2) - \int_{\mathbb{P}^1} [\log|z|^2]\, \phi(DE_2, D\rho_2') \\
=&\int_{\mathbb{P}^1} [\log|u|^2] \Big(\phi\big((i_0^{13})^*(p_{12}^* DE_2, \tau)\big) - \phi\big((i_\infty^{13})^*(p_{12}^* DE_2, \tau)\big) \Big) \\
=&\int_{\mathbb{P}^1} [\log|u|^2] \Big(p_1^*\big(\phi(E_2,\rho_2) - \phi(E_1 \oplus E_3, \rho_1 \oplus \rho_3)\big) \Big) \\
=&\int_{\mathbb{P}^1} [\log|u|^2] \Big(p_1^*\big(\phi(E_2,\rho_2) - \phi(E_1 \oplus E_3, \rho_1 \oplus \rho_3)\big) \Big) \\
=&0.
\end{aligned}
$$

Here, in the last step, we use the fact that $p_1^*(\phi(E_2,\rho_2) - \phi(E_1 \oplus E_3, \rho_1 \oplus \rho_3))$ is a constant form with respect to $\mathbb{P}^1$ and hence its integration with respect to $\log|u|^2$ over $\mathbb{P}^1$ is identically zero. This proves Claim B.1.1.

(B.3) Next we prove Claim B.1.2. By Axiom 1,

$$
\begin{aligned}
&\int_{\mathbb{P}^1} [\log|z|^2]\, d_{M\times\mathbb{P}^1} d^c_{M\times\mathbb{P}^1} \phi_{\mathrm{BC}}(DE., D\rho.) \\
=&\int_{\mathbb{P}^1} [\log|z|^2]\, \phi(DE_2, D\rho_2) - \int_{\mathbb{P}^1} [\log|z|^2]\, \phi(DE_1 \oplus DE_3, D\rho_1 \oplus D\rho_3) \qquad (1.3) \\
=&\int_{\mathbb{P}^1} [\log|z|^2]\, \phi(DE_2, D\rho_2).
\end{aligned}
$$

Here, in the last step, we use the fact that $\phi(DE_1 \oplus DE_3, D\rho_1 \oplus D\rho_3)$ remains the same if we make a change from z to z^{-1}. Therefore, in $\tilde{A}(M)$,

$$
\begin{aligned}
&\phi'_{\mathrm{BC}}(E., \rho.) \\
=&\int_{\mathbb{P}^1} [\log|z|^2]\, \phi(DE_2, D\rho_2) \\
=&\int_{\mathbb{P}^1} [\log|z|^2]\, d_{M\times\mathbb{P}^1} d^c_{M\times\mathbb{P}^1} \phi_{\mathrm{BC}}(DE., D\rho.) \quad \text{(by (1.3))} \\
=&i_0^* \phi_{\mathrm{BC}}(DE., D\rho.) - i_\infty^* \phi_{\mathrm{BC}}(DE., D\rho.) \\
=&\phi_{\mathrm{BC}}(E., \rho.) - i_\infty^* \phi_{\mathrm{BC}}(DE., D\rho.) \quad \text{(by Axiom 2 as } i_0^*(DE., D\rho.) = (E., \rho.)) \\
=&\phi_{\mathrm{BC}}(E., \rho.) - 0 \quad \text{(by Axiom 3 as } i_\infty^*(DE., D\rho.) \text{ splits)} \\
=&\phi_{\mathrm{BC}}(E., \rho.).
\end{aligned}
$$

This completes the proof of Claim B.1.2 and hence the uniqueness of the classical Bott-Chern secondary characteristic classes.

C. The existence of classical Bott-Chern classes

(C.1) In this section, we show that there exists a construction ϕ_{BC} which satisfies Axioms 1, 2 and 3 in (A.4).

To do so, as above, set

$$\phi_{\mathrm{BC}}(E.,\rho.) := \int_{\mathbb{P}^1} [\log|z|^2] \cdot \Big(\phi(DE_2, D\rho_2)\Big).$$

By the fact that $dd^c[\log|z|^2] = \delta_0 - \delta_\infty$, and characteristic forms are d-closed and functorial, (see e.g., Theorem A.3(1) and (2),) $\phi_{\mathrm{BC}}(E.,\rho.)$ satisfies Axiom 1.

(C.2) By the fact that $\phi_{\mathrm{BC}}(E.,\rho.)$ does not depend on $D\rho_2$, i.e., Claim B.1.1, and characteristic forms are functorial, i.e., Theorem A.1.3, $\phi_{\mathrm{BC}}(E.,\rho.)$ satisfies Axiom 2 as well.

(C.3) If $(E.,\rho.)$ splits, we may take $(DE.,D\rho.)$ as the pull-back of $(E.,\rho.)$ via the natural projection to $M \times \mathbb{P}^1$. In this case, $\phi(DE_2, D\rho_2)$ does not depend on z. Hence

$$\phi_{\mathrm{BC}}(E.,\rho.) := \int_{\mathbb{P}^1} [\log|z|^2] \cdot \Big(\phi(DE_2, D\rho_2)\Big) = 0.$$

This shows that $\phi_{\mathrm{BC}}(E.,\rho.)$ satisfies Axiom 3 too. This then ends the proof of the uniqueness and the existence of classical Bott-Chern secondary characteristic classes.

(C.4) We end this chapter with the following remarks. At the very beginning, it is not quite clear why the secondary characteristic classes should be in $\tilde{A}(M)$, a quotient of $A(M)$. One possible explanation is that with the limitation of the methods employed, the uniqueness can only be established after modulo exact forms. On the other hand, we may equally use other spaces as well, e.g., $\hat{A}(M) := A(M)/(\text{exact forms} + d - \text{closed forms})$. If we wish, we may justify this later choice of $\hat{A}(M)$ by arguing that, after all, $\phi(E,\rho)$ is equal to $\phi(E, c\cdot\rho)$ for any positive constant c, yet, in general, $\phi_{\mathrm{BC}}(E;\rho, c\cdot\rho) \neq 0$ in $\tilde{A}(M)$, (so it is too sensitive,) and the d-closed forms have already been taken care of in the theory of characteristic classes (so in the theory of secondary characteristic classes, d-closed forms should be simply viewed as zero). As a matter of fact, later in Section 7.B, we use yet a third space to understand classical Bott-Chern classes.

2. Axioms for relative Bott-Chern Secondary Characteristic Classes

A) Downstairs Rule

B) Projection Rule

C) Functorial Rule

D) Uniqueness Rule w.r.t. Hermitian Bundles

E) Uniqueness Rule w.r.t. Metrized Morphisms

In this chapter, we naturally expose first five of six key axioms for the so-called relative Bott-Chern secondary characteristic classes associated to smooth proper morphisms of compact Kähler manifolds.

A. Downstairs Rule

(A.1) Let X and Y be two compact Kähler manifolds. Let $f : X \to Y$ be a smooth, proper morphism between X and Y. For any $y \in Y$, denote by X_y the fiber of f at y. Fix a hermitian metric τ_f on T_f, the relative tangent vector bundle of f. Moreover, we assume that the metric on X_y induced by the restriction of (T_f, τ_f) is Kähler for all $y \in Y$.

Let (E, ρ) be a hermitian vector bundle on X. Assume that E is f-acyclic, i.e., all higher direct images $R^i f_* E$ vanish, where $i > 0$. As a direct consequence of this assumption, the direct image f_*E of E is a vector bundle on Y. Moreover, f_*E admits a natural metric $L^2(\rho, \tau)$, the L^2-metric induced by taking integration of ρ along f with respect to τ_f. In the sequel, we call $(f : X \to Y; E, \rho; T_f, \tau_f)$, or simply $(E, \rho; f, \tau_f)$ a *properly metrized datum* if all the above conditions are satisfied.

Associated to a properly metrized datum $(E, \rho; f, \tau_f)$ is the so-called *relative Bott-Chern secondary characteristic classes* $\mathrm{ch}_{\mathrm{BC}}(E, \rho; f, \tau_f) \in \widetilde{A}(Y)$ which may be essentially characterized by the following axioms.

(A.2) *Axiom 1.* (**Downstairs Rule**) In $A(Y)$,

$$dd^c \mathrm{ch}_{\mathrm{BC}}(E, \rho; f, \tau_f) = f_*\Big(\mathrm{ch}(E, \rho) \cdot \mathrm{td}(T_f, \tau_f)\Big) - \mathrm{ch}\Big(f_*E, L^2(\rho, \tau)\Big).$$

B. Projection Rule

(B.1) Let (F, ρ_F) be a hermitian vector bundle on Y. Then $E \otimes f^*F$ is f-acyclic as well. (Indeed, we know that $f_*(E \otimes f^*F) = f_*E \otimes F$.) So, $(E \otimes f^*F, \rho \otimes f^*\rho_F; f, \tau_f)$ is again a properly metrized datum. Thus, we also have the relative Bott-Chern secondary characteristic classes $\mathrm{ch}_{\mathrm{BC}}(E \otimes f^*F, \rho \otimes f^*\rho_F; f, \tau_f) \in \tilde{A}(Y)$.

(B.2) *Axiom 2.* (**Projection Rule**) In $\tilde{A}(Y)$,

$$\mathrm{ch}_{\mathrm{BC}}(E \otimes f^*F, \rho \otimes f^*\rho_F; f, \tau_f) = \mathrm{ch}_{\mathrm{BC}}(E, \rho; f, \tau_f) \cdot \mathrm{ch}(F, \rho_F).$$

C. Functorial Rule

(C.1) Let $g : Y' \to Y$ be a morphism between two compact Kähler manifolds. Then we have the Cartesian product

$$\begin{array}{ccc} X \times_Y Y' & \stackrel{g_f}{\to} & X \\ f_g \downarrow & & \downarrow f \\ Y' & \stackrel{g}{\to} & Y. \end{array}$$

Define the pull-back of $(E, \rho; f, \tau_f)$ via g_f by

$$g_f^*(E, \rho; f, \tau_f) = (g_f^*(E, \rho); f_g, g_f^*\tau_f).$$

Easily, we see that

(i) by the f-acyclic condition on E, g_f^*E is f_g-acyclic;

(ii) $g_f^*\tau_f$ gives a hermitian metric on the relative tangent bundle T_{f_g} for the smooth, proper morphism f_g which induces Kähler metrics on its fibers.

Hence, $g_f^*(E, \rho; f, \tau_f)$ is again a properly metrized datum, and we may have the corresponding relative Bott-Chern secondary characteristic classes $\mathrm{ch}_{\mathrm{BC}}\Big(g_f^*(E, \rho; f, \tau_f)\Big) \in \widetilde{A}(Y')$ as well.

(C.2) *Axiom 3.* (**Functorial Rule**) In $\tilde{A}(Y')$,

$$g_f^*\Big(\mathrm{ch}_{\mathrm{BC}}(E, \rho; f, \tau_f)\Big) = \mathrm{ch}_{\mathrm{BC}}\Big(g_f^*(E, \rho; f, \tau_f)\Big).$$

D. Uniqueness Rule w.r.t. Hermitian Bundles

(D.1) Let $E.: 0 \to E_1 \to E_2 \to E_3 \to 0$ be an exact sequence of f-acyclic vector bundles on X. Then the direct image of $E.$ gives an exact sequence of vector bundles on Y as follows;

$$f_*E.: \qquad 0 \to f_*E_1 \to f_*E_2 \to f_*E_3 \to 0.$$

If we put hermitian metrics ρ_i on E_i, $i = 1, 2, 3$, we then get naturally L^2-metrics $L^2(\rho_i, \tau_f)$ on f_*E_i. By Theorem 1.A.5, there exists the so-called (classical) Bott-Chern secondary characteristic classes associated to both $(E., \rho.)$ and $(f_*E., L^2(\rho., \tau_f))$. Denote these two classical Bott-Chern secondary characteristic classes by $\mathrm{ch}_{\mathrm{BC}}(E., \rho.) \in \widetilde{A}(X)$ and $\mathrm{ch}_{\mathrm{BC}}(f_*E., L^2(\rho., \tau_f)) \in \widetilde{A}(Y)$ respectively. Surely, for each (E_i, ρ_i), $i = 1, 2, 3$, we have the corresponding properly metrized datum $(E_i, \rho_i; f, \tau_f)$, and hence the relative Bott-Chern secondary characteristic classes $\mathrm{ch}_{\mathrm{BC}}(E_i, \rho_i; f, \tau_f) \in \widetilde{A}(Y)$.

(D.2) *Axiom 4.* (**Uniqueness Rule w.r.t. Hermitian Bundles**) In $\tilde{A}(Y)$,

$$\begin{aligned}&\mathrm{ch}_{\mathrm{BC}}(E_2, \rho_2; f, \tau_f) - \mathrm{ch}_{\mathrm{BC}}(E_1, \rho_1; f, \tau_f) - \mathrm{ch}_{\mathrm{BC}}(E_3, \rho_3; f, \tau_f)\\ =&f_*\Big(\mathrm{ch}_{\mathrm{BC}}(E., \rho.) \cdot \mathrm{td}(T_f, \tau_f)\Big) - \mathrm{ch}_{\mathrm{BC}}\Big(f_*E., L^2(\rho., \tau_f)\Big).\end{aligned}$$

E. Uniqueness Rule w.r.t. Metrized Morphisms

(E.1) Let Z be a compact Kähler manifold and $g: Y \to Z$ be a smooth, proper morphism. So we have a triangle of smooth fibrations of compact Kähler manifolds among compact Kähler manifolds as follows;

$$\begin{array}{ccc} X & \stackrel{f}{\to} & Y \\ & g\circ f \searrow & \downarrow g \\ & & Z. \end{array}$$

Naturally, we have the short exact sequence of relative tangent bundles

$$T.: \qquad 0 \to T_f \to T_{g\circ f} \to f^*T_g \to 0,$$

where T_f, T_g and $T_{g\circ f}$ stand for the relative tangent bundles associated to the smooth, proper morphisms f, g and $g \circ f$ respectively. Put hermitian metrics τ_f, τ_g and $\tau_{g\circ f}$ on T_f, T_g and $T_{g\circ f}$ respectively such that the induced metrics on all corresponding fibers are Kähler. By Theorem 1.A.5, we have the corresponding

(classical) Bott-Chern secondary characteristic classes $\mathrm{td}_{\mathrm{BC}}(T.,\tau.) \in \widetilde{A}(X)$ with respect to the Todd characteristic class.

(E.2) Recall that E is f-acyclic so that f_*E is a vector bundle on Y. In addition, assume that E is $g\circ f$-acyclic and f_*E is g-acyclic. As a direct consequence, $(g\circ f)_*E$ is a vector bundle on Z and is equal to $g_*(f_*E)$. All this gives us the properly metrized data $(E,\rho;f,\tau_f)$, $(f_*E,L^2(\rho,\tau_f);g,\tau_g)$, and $(E,\rho;g\circ f,\tau_{g\circ f})$, and hence their associated relative Bott-Chern secondary characteristic classes $\mathrm{ch}_{\mathrm{BC}}(E,\rho;f,\tau_f) \in \widetilde{A}(Y)$, $\mathrm{ch}_{\mathrm{BC}}(f_*E,L^2(\rho,\tau_f);g,\tau_g) \in \widetilde{A}(Z)$ and $\mathrm{ch}_{\mathrm{BC}}(E,\rho;g\circ f,\tau_{g\circ f}) \in \widetilde{A}(Z)$.

Moreover, on Z, we have a short exact sequence of vector bundles

$$0 \to g_*(f_*E) \to (g\circ f)_*E \to 0.$$

Put the L^2-metrics $L^2(\rho,\tau_{g\circ f})$ and $L^2(L^2(\rho,\tau_f),\tau_g)$ on $(g\circ f)_*E$ and on $g_*(f_*E)$ respectively, we then have the associated (classical) Bott-Chern secondary characteristic classes, which, as in Remark 1.1.3 in 1.A.5, we denote by

$$\mathrm{ch}_{\mathrm{BC}}\Big((g\circ f)_*E; L^2(\rho,\tau_{g\circ f}), L^2(L^2(\rho,\tau_f),\tau_g)\Big) \in \widetilde{A}(Z).$$

(E.3) *Axiom 5.* (**Uniqueness Rule w.r.t. Metrized Morphisms**) In $\tilde{A}(Z)$,

$$\begin{aligned}&\mathrm{ch}_{\mathrm{BC}}(E,\rho;g\circ f,\tau_{g\circ f})\\ &\qquad - g_*\Big(\mathrm{ch}_{\mathrm{BC}}(E,\rho;f,\tau_f)\cdot \mathrm{td}(T_g,\tau_g)\Big) - \mathrm{ch}_{\mathrm{BC}}\Big(f_*E,L^2(\rho,\tau_f);g,\tau_g\Big)\\ =&(g\circ f)_*\Big(\mathrm{ch}(E,\rho)\cdot \mathrm{td}_{\mathrm{BC}}(T.,\tau.)\Big) - \mathrm{ch}_{\mathrm{BC}}\Big((g\circ f)_*E; L^2(\rho,\tau_{g\circ f}), L^2(L^2(\rho,\tau_f),\tau_g)\Big).\end{aligned}$$

(E.4) To state the last axiom, the deformation to the normal cone rule, we need to make a lengthy discussion. So we delay it untill the next chapter.

3. Deformation to the Normal Cone Rule

A) Deformation to the normal cone: an algebraic construction

B) Deformation to the normal cone: the associated metrics

C) Deformation to the normal cone rule

In this chapter, we give the final axiom for the relative Bott-Chern secondary characteristic classes, the deformation to the normal cone rule.

A. Deformation to the normal cone: an algebraic construction

(A.1) Let $f : X \to Y$ and $g : Z \to Y$ be two smooth, proper morphisms of compact Kähler manifolds and $i : X \to Z$ be a codimension one closed immersion over Y, i.e., i is a closed immersion of codinemsion one such that $f = g \circ i$. Then we have the following standard construction of the deformation to the normal cone.

Denote by

$$\pi : W := B_{X\times\{\infty\}}Z \times \mathbb{P}^1 \to Z \times \mathbb{P}^1,$$

the natural projection, where $B_{X\times\{\infty\}}Z \times \mathbb{P}^1$ denotes the blowing-up of $Z \times \mathbb{P}^1$ along $X \times \{\infty\}$. Denote the exceptional divisor of π by $\mathbb{P}$. It is well-known that the map $q_W : W \to \mathbb{P}^1$, obtained by composing π with the projection $q : Z \times \mathbb{P}^1 \to \mathbb{P}^1$, is flat, and that for $t \in \mathbb{P}^1$:

$$q_W^{-1}(t) = \begin{cases} Z \times \{t\}, & \text{for } t \neq \infty, \\ \mathbb{P} \cup B_X Z, & \text{for } t = \infty. \end{cases}$$

Here $B_X Z$ denotes the blowing-up of Z along X. Moreover, by the construction, $\mathbb{P}$ and $B_X Z$ intersect transversally, and $\mathbb{P} \cap B_X Z$ is the exceptional divisor X on $B_X Z(= Z \times \{\infty\}$, due to the dimensional reason).

Denote by $I : X \times \mathbb{P}^1 \hookrightarrow W$ the induced codimension one closed embedding. Easily we see that the image of I does not intersect with $B_X Z$, and the image $X \times \{\infty\}$ in W is a section of $\mathbb{P}$. Denote the induced fibration $Z \times \{t\} \to Y \times \{t\}$ by g_t for $t \neq \infty$ and set g_∞ to be the composition of the projection of $\mathbb{P}$ on X with $(X =)X \times \{\infty\} \to Y \times \{\infty\}(= Y)$. Denote by $f_t : X \times \{t\} \to Y \times \{t\}$ the smooth morphisms induced from f for all $t \in \mathbb{P}^1$, by i'_∞ the inclusion of $W_\infty = \mathbb{P} + B_X Z$ into W, and by k (resp. l) the inclusion of $\mathbb{P}$ (resp. $B_X Z$) to W. So we have the

following commutative diagram;

$$
\begin{array}{ccccccccc}
X \times \{t\} & & \hookrightarrow & & X \times \mathbb{P}^1 & & \hookleftarrow & & X \times \{\infty\} \\
| & i_t \searrow & & & I \downarrow & & & \swarrow i_\infty & | \\
| & & & & & k \swarrow & \mathbb{P} & & | \\
| & (t \neq \infty) & Z \times \{t\} & \hookrightarrow & B_{X \times \{\infty\}} Z \times \mathbb{P}^1 & \overset{i'_\infty}{\hookleftarrow} & + & \searrow g_\infty & | \\
| & & & & & l \nwarrow & Z & & | \\
f_t & & & \searrow & \pi \downarrow & \swarrow & & & f_\infty \\
| & g_t \swarrow & & & Z \times \mathbb{P}^1 & & & \searrow & | \\
| & & & & \downarrow & & & & | \\
\downarrow & & & & & & & & \downarrow \\
Y \times \{t\} & & \hookrightarrow & & Y \times \mathbb{P}^1 & & \hookleftarrow & & Y \times \{\infty\}
\end{array}
$$

(A.2) Now let E be a g-acyclic vector bundle on Z such that $E(-X) := E \otimes \mathcal{O}_Z(-X)$ is g-acyclic as well. In particular, we have the short exact sequence

$$0 \to E(-X) \to E \to i_* i^* E \to 0 \tag{3.1}$$

on Z. We want to introduce a $\mathbb{P}^1$-deformation for (3.1) via $W \to \mathbb{P}^1$.

First, pulling it back to $Z \times \mathbb{P}^1$, we have the short exact sequence $0 \to p_Z^* E(-X \times \mathbb{P}^1) \to p_Z^* E \to I'_*(I')^* E \to 0$ on $Z \times \mathbb{P}^1$. Here $p_Z : Z \times \mathbb{P}^1 \to Z$ denotes the natural projection and $I' : X \times \mathbb{P}^1 \hookrightarrow Z \times \mathbb{P}^1$ denotes the induced injection from i.

To get a $\mathbb{P}^1$-deformation for (3.1) on W, we should take, instead of simply taking $\pi^*(p_Z^* E(-X \times \mathbb{P}^1))$, the vector bundle $\pi^*(p_Z^* E(Z \times \{\infty\} - X \times \mathbb{P}^1))$. (For a possible motivation, see e.g., the definition DE_1 in 1.B.1.) Then, we know that

$$
\begin{aligned}
&\pi^*(p_Z^* E(Z \times \{\infty\} - X \times \mathbb{P}^1)) \\
=&(\pi \circ p_Z)^* E \otimes \pi^* \mathcal{O}_{Z \times \mathbb{P}^1}(Z \times \{\infty\} - X \times \mathbb{P}^1) \\
=&(\pi \circ p_Z)^* E \otimes \mathcal{O}_W(\mathbb{P} + B_X Z - X \times \mathbb{P}^1 - \mathbb{P}) \\
=&(\pi \circ p_Z)^* E \otimes \mathcal{O}_W(B_X Z - X \times \mathbb{P}^1) \\
=&(\pi \circ p_Z)^* E(B_X Z - X \times \mathbb{P}^1).
\end{aligned}
$$

So the correct choice of the $\mathbb{P}^1$-deformation of (3.1) on W we seek is the following exact sequence of coherent sheaves on W;

$$0 \to (\pi \circ p_Z)^* E(B_X Z - X \times \mathbb{P}^1) \to (\pi \circ p_Z)^* E(B_X Z) \to I_* I^*((\pi \circ p_Z)^* E(B_X Z)) \to 0. \tag{3.2}$$

Indeed, by the flatness of $q_W : W \to \mathbb{P}^1$, the restrictions of (3.2) to the fibers W_t of q_W for all $t \in \mathbb{P}^1$ are exact. Thus, in particular, for each $t \neq \infty$ in $\mathbb{P}^1$, from (3.2), we have the induced exact sequence

$$0 \to E_t(-X) \to E_t \to (i_t)_* i_t^* E_t \to 0 \qquad \text{over} \quad Z \times \{t\} \tag{3.3}$$

as $(\pi \circ p_Z)^* E(B_X Z)\big|_{Z \times \{t\}} = E_t$, $(\pi \circ p_Z)^* E(B_X Z - X \times \mathbb{P}^1)\big|_{Z \times \{t\}} = E_t(-X)$ with E_t the pull-back of E under the canonical identity $X \times \{t\} \simeq X$. Similarly, for the fiber at ∞, if we set $E(B_X Z)\big|_{\mathbb{P}} := E_\infty$, $E(B_X Z)\big|_{B_X Z} =: E'_\infty$ and $E(B_X Z - X \times \mathbb{P}^1)\big|_{B_X Z} =: E''_\infty$. Then $E(B_X Z - X \times \mathbb{P}^1)\big|_{\mathbb{P}} = E_\infty(-X)$, and (3.2) splits into two exact sequences

$$0 \to E_\infty(-X) \to E_\infty \to (i_\infty)_* i_\infty^* E_\infty \to 0 \qquad \text{over} \quad \mathbb{P} \tag{3.4}$$

and

$$0 \to E'_\infty \to E''_\infty \to 0 \to 0 \qquad \text{over} \quad B_X Z. \tag{3.5}$$

Here in the last statement, we use that fact that $I(X \times \mathbb{P})$ is away from $B_X Z$ in W. Thus, in particular, on $B_X Z$, $E'_\infty = E''_\infty$. Now we further assume that E on Z is chosen so that E_∞ and $E_\infty(-X)$ are all g_∞-acyclic.

Note that $X \times \mathbb{P}^1$ is away from $B_X Z$, the restriction of $E_\infty - E_\infty(-X)$ to $B_X Z$ gives a zero element in the K-group of $\mathbb{P} \cap B_X Z$. As a direct consequence, in $K(Y)$, we get, for all $t \in \mathbb{P}^1$,

$$(g_t)_*(E_t) - (g_t)_*(E_t(-X)) = (g_\infty)_*(E_\infty) - (g_\infty)_*(E_\infty(-X)). \tag{3.6}$$

B. Deformation to the normal cone: the associated metrics

(B.1) With the same notation as in the previous section, choose a Kähler metric τ_W on W. Then τ_W naturally induces Kähler metrics τ_{g_t} and τ_{g_∞} on $Z_t := Z \times \{t\}$ and $\mathbb{P}$ respectively for all $t \neq \infty$. Note that for a smooth morphism, the relative tangent bundle is a subbundle of the tangent bundle of the total space, by taking restriction again, we then get hermitian metrics τ_t on T_{g_t} for all $t \in \mathbb{P}^1$. Easily, we see that the induced metrics for fibers of g_t from τ_t are all Kähler.

Now let E be a g-acyclic vector bundles on Z such that E_t and $E_t(-X)$ are g_t-acyclic as well for all $t \in \mathbb{P}^1$. Fix hermitian metrics ρ and ρ' on E and on $E(-X)$ respectively. Use the same notation to denote the pull-back of (E, ρ) onto W. Choose the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}^1}(\infty)$ and a metric on $\mathcal{O}_W(-X \times \mathbb{P}^1)$ such that in a neighborhood U of $B_X Z$, which is away from $X \times \mathbb{P}^1$, the natural isomorphism

$\mathcal{O}_W(-X \times \mathbb{P}^1) \simeq \mathcal{O}_W$ induces an isometry, once we put the trivial metric on $\mathcal{O}_W$. Denote these final induced metrics on $E(B_X Z)$ and $E(B_X Z - X \times \mathbb{P}^1)$ by $D\rho$ and $D\rho'$ respectively. (Question: Figure out how the metric on $\mathcal{O}_W(B_X Z)$ is defined?)

Denote the induced metrics via restrction to E_t and $E_t(-X)$ (resp. to E'_∞ and E''_∞ on $B_X E$) by ρ_t and ρ'_t respectively for all $t \in \mathbb{P}^1$, (resp. ρ''_∞ and ρ'''_∞). Easily, we see that $\rho_0 = \rho$ and $\rho'_0 = \rho'$, and $(E'_\infty, \rho''_\infty)$ is isomorphic to $(E''_\infty, \rho'''_\infty)$ by the construction. In this way,

$$(E(-X), \rho') \hookrightarrow (E, \rho) \qquad \text{on} \quad Z$$

is deformed to

$$(E_\infty(-X), \rho'_\infty) \hookrightarrow (E_\infty, \rho_\infty) \quad \text{on} \quad \mathbb{P}$$

(and

$$(E'_\infty, \rho''_\infty) \simeq (E''_\infty, \rho'''_\infty) \qquad \text{on} \quad B_X Z).$$

In particular, we have, for all $t \in \mathbb{P}^1$,

(i) smooth, proper morphisms g_t for compact Kähler manifolds, together with hermitian metrics τ_t on relative tangent bundles $T_t := T_{g_t}$, whose induced metrics on all fibers of g_t are Kähler as well;

(ii) g_t-acyclic hermitian vector bundles (E_t, ρ_t) and $(E_t(-X), \rho'_t)$.

Thus, as a direct consequence, we have the properly metrized data $(E_t, \rho_t; g_t, \tau_{g_t})$ and $(E_t(-X), \rho'_t; g_t, \tau_{g_t})$ for all $t \in \mathbb{P}^1$, and their associated relative Bott-Chern secondary characteristic classes $\mathrm{ch}_{\mathrm{BC}}(E_t, \rho_t; g_t, \tau_{g_t}) \in \tilde{A}(Y)$ and $\mathrm{ch}_{\mathrm{BC}}(E_t(-X), \rho'_t; g_t, \tau_{g_t}) \in \tilde{A}(Y)$ for all $t \in \mathbb{P}^1$. At this point, we may say that the final axiom for the relative Bott-Chern secondary characteristic classes is the one to understand the relation between

$$\mathrm{ch}_{\mathrm{BC}}(E_0, \rho_0; g_0, \tau_{g_0}) - \mathrm{ch}_{\mathrm{BC}}(E_0(-X), \rho'_0; g_0, \tau_{g_0})$$

and

$$\mathrm{ch}_{\mathrm{BC}}(E_\infty, \rho_\infty; g_\infty, \tau_{g_\infty}) - \mathrm{ch}_{\mathrm{BC}}(E_\infty(-X), \rho'_\infty; g_\infty, \tau_{g_\infty})$$

in $\tilde{A}(Y)$.

(B.2) To facilitate the ensuring discussion, we now find out how $\mathrm{ch}_{\mathrm{BC}}(E_t, \rho_t; g_t, \tau_{g_t}) \in \tilde{A}(Y)$ and $\mathrm{ch}_{\mathrm{BC}}(E_t(-X), \rho'_t; g_t, \tau_{g_t}) \in \tilde{A}(Y)$ change with respect to $t \in \mathbb{A}^1$.

By the construction, for $t \neq \infty$, we may simply view $g_t : Z \times \{t\} \to Y \times \{t\}$ as the original $g : Z \to Y$. In this way, $\mathrm{ch}_{\mathrm{BC}}(E_t, \rho_t; g_t, \tau_{g_t}) \in \tilde{A}(Y)$ may be interpretated as

the relative Bott-Chern secondary characteristic classes for the original morphism g but with metrics ρ_t on E and τ_t on T_g. Moreover, we may get this type of changing metrics in two steps: First, from (ρ_0, τ_0) to (ρ_0, τ_t); then from (ρ_0, τ_t) to (ρ_t, τ_t). In this way, by applying Axiom 5 in step 1 and Axiom 4 in step 2, we have the following relations:

$$\begin{aligned}
&\mathrm{ch}_{\mathrm{BC}}(E_0, \rho_0; g_t, \tau_{g_t}) - \mathrm{ch}_{\mathrm{BC}}(E_0, \rho_0; g_0, \tau_{g_0})\\
=&g_*\Big(\mathrm{ch}(E; \rho_0) \cdot \mathrm{td}_{\mathrm{BC}}(T_g; \tau_t, \tau_0)\Big) - \mathrm{ch}_{\mathrm{BC}}\Big(g_*E; L^2(\rho_0, \tau_t), L^2(\rho_0, \tau_0)\Big),
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{ch}_{\mathrm{BC}}(E_t, \rho_t; g_t, \tau_{g_t}) - \mathrm{ch}_{\mathrm{BC}}(E_0, \rho_0; g_t, \tau_{g_t})\\
=&g_*\Big(\mathrm{ch}_{\mathrm{BC}}(E; \rho_t, \rho_0) \cdot \mathrm{td}(T_g, \tau_t)\Big) - \mathrm{ch}_{\mathrm{BC}}\Big(g_*E; L^2(\rho_t, \tau_t), L^2(\rho_0, \tau_t)\Big).
\end{aligned}$$

That is to say,

$$\begin{aligned}
&\mathrm{ch}_{\mathrm{BC}}(E_t, \rho_t; g_t, \tau_{g_t}) - \mathrm{ch}_{\mathrm{BC}}(E_0, \rho_0; g_0, \tau_{g_0})\\
=&g_*\Big(\mathrm{ch}_{\mathrm{BC}}(E; \rho_t, \rho_0) \cdot \mathrm{td}(T_g, \tau_t) + \mathrm{ch}(E; \rho_0) \cdot \mathrm{td}_{\mathrm{BC}}(T_g; \tau_t, \tau_0)\Big)\\
&- \mathrm{ch}_{\mathrm{BC}}\Big(g_*E; L^2(\rho_t, \tau_t), L^2(\rho_0, \tau_0)\Big).
\end{aligned} \tag{3.7}$$

Similarly,

$$\begin{aligned}
&\mathrm{ch}_{\mathrm{BC}}(E_t(-X), \rho_t'; g_t, \tau_{g_t}) - \mathrm{ch}_{\mathrm{BC}}(E_0(-X), \rho_0'; g_0, \tau_{g_0})\\
=&g_*\Big(\mathrm{ch}_{\mathrm{BC}}(E(-X); \rho_t', \rho_0') \cdot \mathrm{td}(T_g, \tau_t) + \mathrm{ch}(E(-X); \rho_0') \cdot \mathrm{td}_{\mathrm{BC}}(T_g; \tau_t, \tau_0)\Big)\\
&- \mathrm{ch}_{\mathrm{BC}}\Big(g_*(E(-X)); L^2(\rho_t', \tau_t), L^2(\rho_0', \tau_0)\Big).
\end{aligned} \tag{3.8}$$

Therefore,

$$\begin{aligned}
&\mathrm{ch}_{\mathrm{BC}}(E_t, \rho_t; g_t, \tau_{g_t}) - \mathrm{ch}_{\mathrm{BC}}(E_t(-X), \rho_t'; g_t, \tau_{g_t})\\
&\qquad - \Big(\mathrm{ch}_{\mathrm{BC}}(E_0, \rho_0; g_0, \tau_{g_0}) - \mathrm{ch}_{\mathrm{BC}}(E_0(-X), \rho_0'; g_0, \tau_{g_0})\Big)\\
=&g_*\Big((\mathrm{ch}_{\mathrm{BC}}(E; \rho_t, \rho_0) - \mathrm{ch}_{\mathrm{BC}}(E(-X); \rho_t', \rho_0')) \cdot \mathrm{td}(T_g, \tau_t)\\
&\qquad + (\mathrm{ch}(E; \rho_0) - \mathrm{ch}(E(-X); \rho_0')) \cdot \mathrm{td}_{\mathrm{BC}}(T_g; \tau_t, \tau_0)\Big)\\
&- \Big(\mathrm{ch}_{\mathrm{BC}}\Big(g_*E; L^2(\rho_t, \tau_t), L^2(\rho_0, \tau_0)\Big) - \mathrm{ch}_{\mathrm{BC}}\Big(g_*(E(-X)); L^2(\rho_t', \tau_t), L^2(\rho_0', \tau_0)\Big)\Big).
\end{aligned} \tag{3.9}$$

So if $q_W : W \to \mathbb{P}^1$ was smooth, we would then have found that the relation between

$$\mathrm{ch}_{\mathrm{BC}}(E_\infty, \rho_\infty; g_\infty, \tau_{g_\infty}) - \mathrm{ch}_{\mathrm{BC}}(E_\infty(-X), \rho_\infty'; g_\infty, \tau_{g_\infty})$$

and

$$\mathrm{ch}_{\mathrm{BC}}(E_0, \rho_0; g_0, \tau_{g_0}) - \mathrm{ch}_{\mathrm{BC}}(E_0(-X), \rho_0'; g_0, \tau_{g_0})$$

can be deduced from Axioms 4 and 5.

C. Deformation to the normal rule

(C.1) Unfortunately, as indicated before, $q_W : W \to \mathbb{P}^1$ is far from being smooth. So we cannot simply deduce a relation between the relative Bott-Chern secondary characteristic classes for the fibers at 0 and ∞ directly by only using previous 5 axioms for relative Bott-Chern secondary characteristic classes with respect to smooth morphisms. Nevertheless, it seems most likely that with a suitable generalization of relative Bott-Chern classes, say for proper and flat morphisms, one then can understand how the relative Bott-Chern classes change even across ∞. All this is indeed the motivation for introducing the so-called ternary characteristic classes in [We2], which themselves need an additional work to justify. So here, we take an alternative approach by sticking on smooth, proper morphisms.

(C.2) The idea is also quite simple. That is, instead of trying to find out what is exactly the difference between

$$\mathrm{ch}_{\mathrm{BC}}(E_\infty, \rho_\infty; g_\infty, \tau_{g_\infty}) - \mathrm{ch}_{\mathrm{BC}}(E_\infty(-X), \rho'_\infty; g_\infty, \tau_{g_\infty})$$

and

$$\mathrm{ch}_{\mathrm{BC}}(E_0, \rho_0; g_0, \tau_{g_0}) - \mathrm{ch}_{\mathrm{BC}}(E_0(-X), \rho'_0; g_0, \tau_{g_0}),$$

we view $\{\mathrm{ch}_{\mathrm{BC}}(E_t, \rho_t; g_t, \tau_{g_t}) - \mathrm{ch}_{\mathrm{BC}}(E_t(-X), \rho'_t; g_t, \tau_{g_t})\}_{t\in\mathbb{P}^1}$ as a family on $\mathbb{P}^1$ and try to establish the continuity of such a family of forms, or better of classes.

It looks that we are now in a position to state the final axiom. No, it is not so. There is yet another difficulty: It is well-known that for the family $W \to Y \times \mathbb{P}^1$, usually the induced L^2-metrics on the direct images for g_t when $t \to \infty$ have singularities. So we need to take care of such singularities as well. (Recall that by Axiom 1 in 2.A.2 for relative Bott-Chern secondary characteristic classes,

$$dd^c\mathrm{ch}_{\mathrm{BC}}(E_t, \rho_t; g_t, \tau_{g_t}) = (g_t)_*\Big(\mathrm{ch}(E_t, \rho_t)\cdot \mathrm{td}(T_t, \tau_{g_t})\Big) - \mathrm{ch}\Big((g_t)_*E, L^2(\rho_t, \tau_{g_t})\Big),$$

and

$$\begin{aligned} &dd^c\mathrm{ch}_{\mathrm{BC}}(E_t(-X), \rho'_t; g_t, \tau_{g_t})\\ &\qquad = (g_t)_*\Big(\mathrm{ch}(E_t(-X), \rho_t)\cdot \mathrm{td}(T_t, \tau_{g_t})\Big) - \mathrm{ch}\Big((g_t)_*E(-X), L^2(\rho'_t, \tau_{g_t})\Big).) \end{aligned}$$

For this purpose, consider the virtual vector bundle $(g_t)_*(E_t) - (g_t)_*(E_t(-X))$ on Y for $t \in \mathbb{P}^1$. By the discussion in (B.1), in particular, (3.6), we know that for

$t \neq \infty$, $(g_t)_*(E_t) = (g_0)_*(E_0)$, $(g_t)_*(E_t(-X)) = (g_0)_*(E_0(-X))$ once we identify $Y \times \{0\}$ with $Y \times \{t\}$, and as virtual vector bundles on Y,

$$(g_\infty)_*(E_\infty) - (g_\infty)_*(E_\infty(-X)) = (g_t)_*(E_t) - (g_t)_*(E_t(-X)).$$

So in particular, there exists a natural number r such that, locally over Y, as holomorphic vector bundles, we have the local isomorphism

$$\mathcal{O}_Y^{\oplus r} \oplus (g_\infty)_*(E_\infty) \oplus (g_t)_*(E_t(-X)) = \mathcal{O}_Y^{\oplus r} \oplus (g_t)_*(E_t) \oplus (g_\infty)_*(E_\infty(-X)).$$

Hence, by the fact that Chern form depends only on the C^∞-structure of a vector bundle, we may choose hermitian metrics γ_t and γ_t' on $(g_t)_*E$ and $(g_t)_*(E_t(-X))$ respectively for all $t \in \mathbb{P}^1$ such that

$$\mathrm{ch}(E_t, \gamma_t) - \mathrm{ch}(E_t(-X), \gamma_t') = \mathrm{ch}(E_\infty, \gamma_\infty) - \mathrm{ch}(E_\infty(-X), \gamma_\infty'). \tag{3.10}$$

With this, via $Y \times \{0\} = Y \times \{\infty\}$, we have the classical Bott-Chern secondary characteristic classes $\mathrm{ch}_{\mathrm{BC}}((g_t)_*(E_t; L^2(\rho_t, \tau_t), \gamma_t)$ and $\mathrm{ch}_{\mathrm{BC}}((g_t)_*(E_t(-X)); L^2(\rho_t', \tau_t), \gamma_t')$. Now we may consider the family

$$\begin{aligned}&\{\Big(\mathrm{ch}_{\mathrm{BC}}(E_t, \rho_t; g_t, \tau_{g_t}) - \mathrm{ch}_{\mathrm{BC}}(E_t(-X), \rho_t'; g_t, \tau_{g_t})\Big)\\ &+ \Big(\mathrm{ch}_{\mathrm{BC}}((g_t)_*(E_t); L^2(\rho_t, \tau_t), \gamma_t) - \mathrm{ch}_{\mathrm{BC}}((g_t)_*(E_t(-X)); L^2(\rho_t', \tau_t), \gamma_t')\Big)\}_{t \in \mathbb{P}^1}.\end{aligned}$$

As a matter of fact, the final axiom for the relative Bott-Chern secondary characteristic classes requires that this family is continuous.

(C.3) *Axiom 6.* (**Deformation to the normal cone rule**) In $\tilde{A}(Y)$,

$$\begin{aligned}&\lim_{t \to \infty} \Big(\big(\mathrm{ch}_{\mathrm{BC}}(E_t, \rho_t; g_t, \tau_{g_t}) - \mathrm{ch}_{\mathrm{BC}}(E_t(-X), \rho_t'; g_t, \tau_{g_t})\big)\\ &+ \big(\mathrm{ch}_{\mathrm{BC}}((g_t)_*(E_t); L^2(\rho_t, \tau_t), \gamma_t) - \mathrm{ch}_{\mathrm{BC}}((g_t)_*(E_t(-X)); L^2(\rho_t', \tau_t), \gamma_t')\big)\Big)\\ =&\big(\mathrm{ch}_{\mathrm{BC}}(E_\infty, \rho_\infty; g_\infty, \tau_{g_\infty}) - \mathrm{ch}_{\mathrm{BC}}(E_\infty(-X), \rho_\infty'; g_\infty, \tau_{g_\infty})\big) +\\ &\big(\mathrm{ch}_{\mathrm{BC}}((g_\infty)_*(E_\infty); L^2(\rho_\infty, \tau_\infty), \gamma_\infty) - \mathrm{ch}_{\mathrm{BC}}((g_\infty)_*(E_\infty(-X)); L^2(\rho_\infty', \tau_\infty), \gamma_\infty')\big).\end{aligned}$$

(C.4) All this then gives us the six key axioms for the relative Bott-Chern secondary characteristic classes. The reader may see that Axioms 2 and 3 (resp. Axioms 4 and 5) may be grouped together naturally. So there are essentially four groups of axioms. The first group, the downstairs rule, relates the secondary theory with its origin; the second group tells us how to deal with the changes from the base; the third group tells us how to deal with the changes from the total space; while the last tells us how to relate the theory for different relative dimensions naturally.

4. Uniqueness of Relative Bott-Chern Secondary Characteristic Classes

A) Weak Uniqueness Theorem

B) Strong Uniqueness Theorem

In this chapter, we will state the uniqueness theorem for relative Bott-Chern secondary characteristic classes. There are two versions of it. Roughly speaking, the first uniqueness theorem claims that after modulo d-closed forms, the classes of the relative Bott-Chern secondary characteristic classes are unique, while the second uniqueness theorem claims that after modulo exact forms, the relative Bott-Chern secondary characteristic classes are essentially unique, i.e., the difference of any two possible relative Bott-Chern secondary characteristic classes can be understood precisely.

A. Weak Uniqueness Theorem

(A.1) We have already stated the axioms for relative Bott-Chern secondary characteristic classes associated to smooth, proper metrized morphisms and relative acyclic hermitian vector bundles in Chapter 2. Now we state the first uniqueness result for them. Recall that $(f : X \to Y; E, \rho; T_f, \tau_f)$ is called a properly metrized datum if $f : X \to Y$ is a smooth morphism of compact Kähler manifolds, (E, ρ) is an f-acyclic hermitian vector bundle, and τ_f is a hermitian metric on the relative tangent bundle T_f of f such that the induced metrics on all fibers of f are Kähler.

(A.2) **Theorem.** (**The Weak Uniqueness for Relative Bott-Chern Secondary Characteristic Classes**) *Suppose that there are two constructions ch_{BC} and ch'_{BC} which satisfy six axioms for relative Bott-Chern secondary characteristic classes, then, for all properly metrized data $(f : X \to Y; E, \rho; T_f, \tau_f)$, the images of $ch_{\mathrm{BC}}(E, \rho; f, \tau_f)$ and $ch'_{\mathrm{BC}}(E, \rho; f, \tau_f)$ in $\hat{A}(Y) := A(Y)/(\text{exact forms} + d-\text{closed forms})$ are the same.*

B. Strong Uniqueness Theorem

(B.1) To state the second uniqueness theorem, we need a preparation.

Let B be a subring of $\mathbb{R}$, and let $P(x) \in B[[x]]$ be any symmetric power series. Then for any vector bundle E on Y, by the splitting principle for vector bundles, there exists a unique additive characteristic class $P(E) \in H^*(X, \mathbb{R})$, the de Rham cohomology of X.

Now assume that there exists a construction $\mathrm{ch}_{\mathrm{BC}}$ such that it satisfies the six axioms for relative Bott-Chern secondary characteristic classes. Then, for

any properly metrized datum $(f : X \to Y; E, \rho; T_f, \tau_f)$, we have an element $\mathrm{ch}_{\mathrm{BC}}(E, \rho; f, \tau_f) \in \tilde{A}(Y)$.

With such an existence, then we claim that $\mathrm{ch}_{\mathrm{BC}}(E, \rho; f, \tau_f) + f_*\Big(\mathrm{ch}(E) \cdot \mathrm{td}(T_f) \cdot P(T_f)\Big) \in \tilde{A}(Y)$ for any fixed additive characteristic class P as above also satisfies the listed six axioms for relative Bott-Chern secondary characteristic classes. Indeed, easily, by the fact that $f_*\Big(\mathrm{ch}(E) \cdot \mathrm{td}(T_f) \cdot P(T_f)\Big)$ is in $H^*(Y, \mathbb{R})$, we see that $\mathrm{ch}_{\mathrm{BC}}(E, \rho; f, \tau_f) + f_*\Big(\mathrm{ch}(E) \cdot \mathrm{td}(T_f) \cdot P(T_f)\Big) \in \tilde{A}(Y)$ satisfies the first five axioms. Moreover note that the deformation to the normal cone does not change the cohomology classes, which is the key property used in the proof of Grothendieck-Riemann-Roch theorem in algebraic geometry, we equally can check the axiom 6 for $\mathrm{ch}_{\mathrm{BC}}(E, \rho; f, \tau_f) + f_*\Big(\mathrm{ch}(E) \cdot \mathrm{td}(T_f) \cdot P(T_f)\Big)$. (See e.g., Lemma 5.B.7.4 below.) In this sense, we may say that there is no uniqueness for relative Bott-Chern secondary characteristic classes.

Even the latest statement is absolutely correct, we still can understand precisely the structure of relative Bott-Chern secondary characteristic classes. Roughly speaking, we may say that the twisting by an additive characteristic class stated above is the only flaw for establishing the uniqueness for relative Bott-Chern secondary characteristic classes.

(B.2) **Theorem.** (**The Srong Uniqueness for Relative Bott-Chern Secondary Characteristic Classes**) *Suppose that there are two constructions* ch_{BC} *and* ch'_{BC} *which satisfy six axioms for relative Bott-Chern secondary characteristic classes, then, there exists an additive characteristic class* R *such that, for all properly metrized data* $(f : X \to Y; E, \rho; T_f, \tau_f)$*, in* $\tilde{A}(Y)$*,*

$$ch'_{\mathrm{BC}}(E, \rho; f, \tau_f) = ch_{\mathrm{BC}}(E, \rho; f, \tau_f) + f_*\Big(ch(E) \cdot td(T_f) \cdot R(T_f)\Big).$$

Remark 4.1. The weak uniqueness theorem says that the relative Bott-Chern classes are unique modulo d-closed forms, while the strong uniqueness theorem says that the relative Bott-Chern classes are unique up to a certain well-structured topological term.

Obviously, the weak uniqueness is a direct consequence of the strong uniqueness, so it is sufficient to prove the later one. We will do it in the following two chapters.

5. Some intermediate results

A) Statements of intermediate results

B) The proofs

In this chapter, we prove some intermediate results for relative Bott-Chern secondary characteristic classes, which will be used in the proof of the uniqueness theorems. So in Chapter 5 and Chapter 6, we will assume that there is a construction $\mathrm{ch}_{\mathrm{BC}}$ which satisfies all six axioms for the relative Bott-Chern secondary characteristic classes.

A. Statements of intermediate results

(A.1) Suppose that for any properly metrized datum $(f : X \to Y; E, \rho; T_f, \tau_f)$, there exist elements $\mathrm{ch}_{\mathrm{BC}}(E, \rho; f, \tau_f) \in \tilde{A}(Y)$ and $\mathrm{ch}'_{\mathrm{BC}}(E, \rho; f, \tau_f) \in \tilde{A}(Y)$ which satisfy the six axioms stated in Chapters 2 and 3. Let P be an additive characteristic class and set

$$\mathrm{Err}(E, \rho; f, \tau_f; P) := \mathrm{ch}_{\mathrm{BC}}(E, \rho; f, \tau_f) - \mathrm{ch}'_{\mathrm{BC}}(E, \rho; f, \tau_f) + f_*\Big(\mathrm{ch}(E)\cdot\mathrm{td}(T_f)\cdot P(T_f)\Big).$$

We want to show that there exists a unique universal additive characteristic class R such that $\mathrm{Err}(E, \rho; f, \tau_f; R) = 0$ for all properly metrized data $(E, \rho; f, \tau_f)$. For this purpose, let us state some intermediate results.

First we study how Err depends on hermitian metrics on vector bundles.

Proposition. *Let $f : X \to Y$ be a smooth, proper morphism of smooth complex Kähler manifolds. Fix a hermitian metric τ_f on the relative tangent vector bundle T_f of f such that the induced metrics on all fibers of f are Kähler. Then for any short exact sequence of f-acyclic hermitian vector bundles*

$$E. : 0 \to E_1 \to E_2 \to E_3 \to 0,$$

with hermitian metrics ρ_i on E_i for $i = 1, 2, 3$,

$$\mathit{Err}(E_1, \rho_1; f, \rho_f; P) + \mathit{Err}(E_3, \rho_3; f, \rho_f; P) = \mathit{Err}(E_2, \rho_2; f, \rho_f; P).$$

In particular, $\mathit{Err}(E, \rho; f, \rho_f; P)$ does not depend on the choice of the metric ρ. Furthermore, $\mathit{Err}(E, \rho; f, \rho_f; P)$ lies in the classes of dd^c-closed forms.

(A.2) Now we study how Err depends on hermitian metrics on manifolds.

Proposition. *Let $f : X \to Y$ and $g : Y \to Z$ be two smooth, proper morphisms of compact Kähler manifolds which admit hermitian metrics τ_f, τ_g and $\tau_{g\circ f}$ on the*

relative tangent vector bundles of f, g and $g \circ f$ respectively, such that the induced metrics on all fibers are Kähler. Let (E, ρ) be an f- and $g \circ f$-acyclic hermitian vector bundle on X such that $f_* E$ is g-acyclic as well. Then

$$Err(E, \rho; g \circ f, \tau_{g \circ f}; P) = Err(f_* E, f_* \rho; g, \tau_g; P) + g_*(Err(E, \rho; f, \tau_f; P) \cdot td(T_g, \tau_g)).$$

In particular, $Err(E, \rho; f, \tau_f; P)$ does not depend on the metric τ_f.

Remark 5.1. Because of these two propositions, we from now on in this section denote $\mathrm{Err}(E, \rho; f, \tau_f; P)$ simply by $\mathrm{Err}(E; f; P)$.

(A.3) We may go slight further. Indeed, we have the following

Proposition. *Let $f : X \to Y$ be a smooth, proper morphism of compact Kähler manifolds. There is a natural morphism*

$$Err(\cdot; f, P) : K(X) \to H^*(Y, \mathbb{R}),$$

such that $Err(E; f, P) = Err(E; f; P)$ for all f-acyclic vector bundles E on X.

Remark 5.2. Because of this prosition, we may equally write $\mathrm{Err}(\cdot; f, P)$ as $\mathrm{Err}(\cdot; f; P)$.

(A.4) We have a functorial property as well.

Proposition. *Let $f : X \to Y$ be a smooth, proper morphism of compact Kähler manifolds. Then, for the Cartesian diagram*

$$\begin{array}{ccc} Y' \times_Y X & \stackrel{g_f}{\to} & X \\ f_g \downarrow & & \downarrow f \\ Y' & \stackrel{g}{\to} & Y \end{array}$$

induced from a base change $g : Y' \to Y$ of compact Kähler manifolds,

$$g^* Err(E; f; P) = Err(g_f^* E; f_g; P).$$

(A.5) We now give a projection formula for Err.

Proposition. *Let $f : X \to Y$ be a smooth, proper morphism of compact Kähler manifolds. Let E and F be vector bundles on X and Y respectively. Then*

$$Err(E \otimes f^* F; f; P) = Err(E; f; P) \cdot ch(F).$$

(A.6) We now study what happens when f is a $\mathbb{P}^1$-bundle on Y.

Proposition. *There is a unique characteristic class R for vector bundles of rank ≤ 2 such that for any $\mathbb{P}^1$-bundle $f : X = \mathbb{P}_Y(F) \to Y$, $Err(\cdot; f; R) \equiv 0$.*

(A.7) Finally, we consider Err for closed immersions. For doing so, we introduce a new Err term: Let $i : X \hookrightarrow Z$ be a closed immersion with the smooth structure morphisms $f : X \to Y$ and $g : Z \to Y$ of compact Kähler manifolds, set

$$\mathrm{Err}(E; i; P) := \mathrm{Err}(E; f; P) - \mathrm{Err}(i_* E; g; P).$$

By Proposition A.3, this definition makes sense, even though $i_* E$ is usually a coherent sheaf only.

Proposition. *Let $i : X \hookrightarrow Z$ be a codimension-one regular closed immersion of compact Kähler manifolds over a compact Kähler manifold Y with smooth structure morphisms $f : X \to Y$ and $g : Z \to Y$. Then for any vector bundle E on X,*

$$Err(E; i; P) = 0,$$

for any additive characteristic class P, which, for rank two vector bundles, coincides with R in Proposition A.6.

B. The Proofs

(B.1) *Proof of Proposition (A.1).* In fact, in $\tilde{A}(Y)$,

$$\begin{aligned}
&\mathrm{Err}(E_2, \rho_2; f, \tau_f; P) - \Big(\mathrm{Err}(E_1, \rho_1; f, \tau_f; P) + \mathrm{Err}(E_3, \rho_3; f, \tau_f; P)\Big)\\
=&\mathrm{ch}_{\mathrm{BC}}(E_2, \rho_2; f, \tau_f) - \mathrm{ch}'_{\mathrm{BC}}(E_2, \rho_2; f, \tau_f) + f_*\big(\mathrm{ch}(E_2) \cdot \mathrm{td}(T_f) \cdot P(T_f)\big)\\
&- \Big(\mathrm{ch}_{\mathrm{BC}}(E_1, \rho_1; f, \tau_f) - \mathrm{ch}'_{\mathrm{BC}}(E_1, \rho_1; f, \tau_f) + f_*\big(\mathrm{ch}(E_1) \cdot \mathrm{td}(T_f) \cdot P(T_f)\big)\\
&+ \mathrm{ch}_{\mathrm{BC}}(E_3, \rho_3; f, \tau_f) - \mathrm{ch}'_{\mathrm{BC}}(E_3, \rho_3; f, \tau_f) + f_*\big(\mathrm{ch}(E_3) \cdot \mathrm{td}(T_f) \cdot P(T_f)\big)\Big)\\
&\qquad \text{(by definition)}
\end{aligned}$$

$$\begin{aligned}
=&\mathrm{ch}_{\mathrm{BC}}(E_2, \rho_2; f, \tau_f) - \Big(\mathrm{ch}_{\mathrm{BC}}(E_1, \rho_1; f, \tau_f) + \mathrm{ch}_{\mathrm{BC}}(E_3, \rho_3; f, \tau_f)\Big)\\
&- \Big(\mathrm{ch}'_{\mathrm{BC}}(E_2, \rho_2; f, \tau_f) - \big(\mathrm{ch}'_{\mathrm{BC}}(E_1, \rho_1; f, \tau_f) + \mathrm{ch}'_{\mathrm{BC}}(E_3, \rho_3; f, \tau_f)\big)\Big)\\
&+ f_*\Big(\big(\mathrm{ch}(E_2) - \mathrm{ch}(E_1) - \mathrm{ch}(E_3)\big) \cdot \mathrm{td}(T_f) \cdot P(T_f)\Big)\\
=&f_*\Big(\mathrm{ch}_{\mathrm{BC}}(E., \rho.) \cdot \mathrm{td}(T_f, \tau_f)\Big) - \mathrm{ch}_{\mathrm{BC}}(f_*(E.), L^2(\rho.; \tau_f))\\
&- \Big(f_*\Big(\mathrm{ch}_{\mathrm{BC}}(E., \rho.) \cdot \mathrm{td}(T_f, \tau_f)\Big) - \mathrm{ch}_{\mathrm{BC}}(f_*(E.), L^2(\rho.; \tau_f))\Big)\\
&+ f_*\Big(0 \cdot \mathrm{td}(T_f) \cdot P(T_f)\Big) \qquad \text{(by Axiom 4 in 2.D.2)}\\
=&0.
\end{aligned}$$

This is simply the first statement of the proposition. With this, by taking E_3 to be zero, we get the second statement. Finally,

$$
\begin{aligned}
& dd^c \mathrm{Err}(E,\rho;f,\tau_f;P)\\
=& dd^c\Big(\mathrm{ch}_{\mathrm{BC}}(E,\rho;f,\tau_f) - \mathrm{ch}'_{\mathrm{BC}}(E,\rho;f,\tau_f) + f_*\big(\mathrm{ch}(E_2)\cdot \mathrm{td}(T_f)\cdot P(T_f)\big)\Big)\\
& \qquad \text{(by definition)}\\
=& dd^c\Big(\mathrm{ch}_{\mathrm{BC}}(E,\rho;f,\tau_f)\Big) - dd^c\Big(\mathrm{ch}'_{\mathrm{BC}}(E,\rho;f,\tau_f)\Big) + dd^c\Big(f_*\big(\mathrm{ch}(E_2)\cdot \mathrm{td}(T_f)\cdot P(T_f)\big)\Big)\\
=& f_*\big(\mathrm{ch}(E,\rho)\cdot \mathrm{td}(T_f,\tau_f)\big) - \mathrm{ch}(f_*E, L^2(\rho;\tau_f))\\
& -\Big(f_*\big(\mathrm{ch}(E,\rho)\cdot \mathrm{td}(T_f,\tau_f)\big) - \mathrm{ch}(f_*E, L^2(\rho;\tau_f))\Big) + 0 \quad \text{(by Axiom 1 in 2.A.2)}\\
=& 0.
\end{aligned}
$$

This completes the proof of Proposition A.1.

(B.2) *Proof of Proposition A.2.* Setting $g = \mathrm{Id}_Y$, the identity map of Y, we find that the second statement of this proposition is a consequence of the first one.

Now we prove the first statement.

$$
\begin{aligned}
& \mathrm{Err}(E,\rho;g\circ f,\tau_{g\circ f};P)\\
& \qquad -\Big(\mathrm{Err}(f_*E,f_*\rho;g,\tau_g;P) + g_*(\mathrm{Err}(E,\rho;f,\tau_f;P)\mathrm{td}(T_g,\tau_g))\Big)\\
=& \mathrm{ch}_{\mathrm{BC}}(E,\rho;g\circ f,\tau_{g\circ f}) - \mathrm{ch}'_{\mathrm{BC}}(E,\rho;g\circ f,\tau_{g\circ f})\\
& \qquad + (g\circ f)_*\Big(\mathrm{ch}(E)\cdot \mathrm{td}(T_{g\circ f})\cdot P(T_{g\circ f})\Big)\\
& -\Big(\mathrm{ch}_{\mathrm{BC}}(f_*E, L^2(\rho,\tau_f);g,\tau_g) - \mathrm{ch}'_{\mathrm{BC}}(f_*E, L^2(\rho,\tau_f);g,\tau_g)\\
& \qquad + f_*\Big(\mathrm{ch}(f_*E)\cdot \mathrm{td}(T_g)\cdot P(T_g)\Big)\Big)\\
& - g_*\Big((\mathrm{ch}_{\mathrm{BC}}(E,\rho;f,\tau_f) - \mathrm{ch}'_{\mathrm{BC}}(E,\rho;f,\tau_f)\\
& \qquad + f_*\Big(\mathrm{ch}(E)\cdot \mathrm{td}(T_f)\cdot P(T_f)\Big)) \cdot \mathrm{td}(T_g,\tau_g)\Big) \quad \text{(by definition)}
\end{aligned}
$$

$$
\begin{aligned}
=&\Big(\mathrm{ch}_{\mathrm{BC}}(E,\rho;g\circ f,\tau_{g\circ f})-\mathrm{ch}_{\mathrm{BC}}(f_*E,L^2(\rho,\tau_f);g,\tau_g)\\
&\qquad -g_*\Big(\mathrm{ch}_{\mathrm{BC}}(E,\rho;f,\tau_f)\cdot\mathrm{td}(T_g,\tau_g)\Big)\\
&-\Big(\mathrm{ch}'_{\mathrm{BC}}(E,\rho;g\circ f,\tau_{g\circ f})-\mathrm{ch}'_{\mathrm{BC}}(f_*E,L^2(\rho,\tau_f);g,\tau_g)\\
&\qquad -g_*\Big(\mathrm{ch}'_{\mathrm{BC}}(E,\rho;f,\tau_f)\cdot\mathrm{td}(T_g,\tau_g)\Big)\\
&+\Big((g\circ f)_*\Big(\mathrm{ch}(E)\cdot\mathrm{td}(T_{g\circ f})\cdot P(T_{g\circ f})\Big)-g_*\Big(\mathrm{ch}(f_*E)\cdot\mathrm{td}(T_g)\cdot P(T_g)\Big)\\
&\qquad -g_*\Big(f_*\Big(\mathrm{ch}(E)\cdot\mathrm{td}(T_f)\cdot P(T_f)\Big)\cdot\mathrm{td}(T_g,\tau_g)\Big)\\
=&(g\circ f)_*\Big(\mathrm{ch}(E,\rho)\cdot\mathrm{td}_{\mathrm{BC}}(T.,\tau.)\Big)\\
&\qquad -\mathrm{ch}_{\mathrm{BC}}\Big((g\circ f)_*E;L^2(\rho,\tau_{g\circ f}),L^2(L^2(\rho,\tau_f),\tau_g)\Big)\\
&-\Big((g\circ f)_*\Big(\mathrm{ch}(E,\rho)\cdot\mathrm{td}_{\mathrm{BC}}(T.,\tau.)\Big)\\
&\qquad -\mathrm{ch}_{\mathrm{BC}}\Big((g\circ f)_*E;L^2(\rho,\tau_{g\circ f}),L^2(L^2(\rho,\tau_f),\tau_g)\Big)\Big)\\
&+\Big((g\circ f)_*\Big(\mathrm{ch}(E)\cdot\mathrm{td}(T_{g\circ f})\cdot P(T_{g\circ f})\Big)-g_*\Big(\mathrm{ch}(f_*E)\cdot\mathrm{td}(T_g)\cdot P(T_g)\Big)\\
&\qquad -g_*\Big(f_*\Big(\mathrm{ch}(E)\cdot\mathrm{td}(T_f)\cdot P(T_f)\Big)\cdot\mathrm{td}(T_g,\tau_g)\Big)\\
&\qquad \text{(by Axiom 4 in 2.D.2)}\\
=&0+(g\circ f)_*\Big(\mathrm{ch}(E)\cdot\mathrm{td}(T_{g\circ f})\cdot P(T_{g\circ f})\Big)-g_*\Big(\mathrm{ch}(f_*E)\cdot\mathrm{td}(T_g)\cdot P(T_g)\Big)\\
&\qquad -g_*\Big(f_*\Big(\mathrm{ch}(E)\cdot\mathrm{td}(T_f)\cdot P(T_f)\Big)\cdot\mathrm{td}(T_g)\Big).
\end{aligned}
$$

From here it is sufficient to prove the following;

Lemma. *With the same notation as above,*

$$
\begin{aligned}
&(g\circ f)_*\Big(ch(E)\cdot td(T_{g\circ f})\cdot P(T_{g\circ f})\Big)\\
&\qquad =g_*\Big(ch(f_*E)\cdot td(T_g)\cdot P(T_g)\Big)+g_*\Big(f_*\Big(ch(E)\cdot td(T_f)\cdot P(T_f)\Big)\cdot td(T_g)\Big).
\end{aligned}
$$

Proof. By definition, P is additive and td is multiplicative. Hence applying them to the exact sequence of relative tangent bundles

$$0\to T_f\to T_{g\circ f}\to f^*T_g\to 0,$$

we have $P(T_{g\circ f})=P(T_f)+f^*P(T_g),\quad \mathrm{td}(T_{g\circ f})=\mathrm{td}(T_f)\cdot f^*\mathrm{td}(T_g)$. Here we use

the fact that as characteristic classes, P and td have the functorial property. Hence

$$\begin{aligned}
&(g\circ f)_*\Big(\mathrm{ch}(E)\cdot \mathrm{td}(T_{g\circ f})\cdot P(T_{g\circ f})\Big)\\
=&(g\circ f)_*\Big(\mathrm{ch}(E)\cdot \mathrm{td}(T_f)\cdot f^*\mathrm{td}(T_g)\cdot \big(P(T_f)+f^*P(T_g)\big)\Big)\\
=&(g\circ f)_*\Big(\mathrm{ch}(E)\cdot \mathrm{td}(T_f)\cdot f^*\mathrm{td}(T_g)\cdot P(T_f)\Big)\\
&\qquad+(g\circ f)_*\Big(\mathrm{ch}(E)\cdot \mathrm{td}(T_f)\cdot f^*\mathrm{td}(T_g)\cdot f^*\big(P(T_g)\big)\Big)\\
=&g_*\Big(f_*\Big(\mathrm{ch}(E)\cdot \mathrm{td}(T_f)\cdot P(T_f)\cdot f^*\mathrm{td}(T_g)\Big)\Big)\\
&\qquad+g_*\Big(f_*\Big(\mathrm{ch}(E)\cdot \mathrm{td}(T_f)\cdot f^*\big(\mathrm{td}(T_g)\cdot P(T_g)\big)\Big)\Big)\\
=&g_*\Big(f_*\Big(\mathrm{ch}(E)\cdot \mathrm{td}(T_f)\cdot P(T_f)\Big)\cdot \mathrm{td}(T_g)\Big)\\
&\qquad+g_*\Big(f_*\Big(\mathrm{ch}(E)\cdot \mathrm{td}(T_f)\Big)\cdot \mathrm{td}(T_g)\cdot P(T_g)\Big)\\
&\qquad\text{(by the projection formula)}\\
=&g_*\Big(f_*\Big(\mathrm{ch}(E)\cdot \mathrm{td}(T_f)\cdot P(T_f)\Big)\cdot \mathrm{td}(T_g)\Big)+g_*\Big(\mathrm{ch}(f_*(E))\cdot \mathrm{td}(T_g)\cdot P(T_g)\Big)\\
&\qquad\text{(by the Grothendieck-Riemann-Roch Theorem in Algebraic Geometry).}
\end{aligned}$$

This completes the proof of the lemma ane hence Proposition A.2.

(B.3) *Proof of Proposition A.3.* The first point we need to address is that by Proposition A.1, Err is only dd^c-closed, but in this proposition, we instead choose the de Rham cohomology groups as the range. This may be solved as follows.

For a compact Kähler manifold Y, denote the space of dd^c-closed forms by $A^*_{dd^c}(Y)$ and set $H^*_{dd^c}(Y,\mathbb{R})$ to be the quotient space of $A^*_{dd^c}(Y)$ modulo the $\partial-$ and $\bar\partial$-forms. It is well-known that for a compact Kähler manifold, a form ω is d-closed and is either $d-$, or $\partial-$, or $\bar\partial$-exact if and only if there exists a form γ such that $dd^c\gamma=\omega$. (See e.g. 7.A.1 later.) Hence if ω is dd^c-closed, then we may choose γ to be $\phi+\partial\alpha+\bar\partial\beta$ with ϕ harmonic. Therefore, we see that $H^*_{dd^c}(Y,\mathbb{R})$ is isomorphic to the cohomology of Y.

The second point here is that we should construct a map Err from the Grothendieck K-group of coherent sheaves of X to $H_{dd^c}(Y,\mathbb{R})$ which coincides with the previous Err for f-acyclic vector bundles. But this is rather formal.

(i) For complex Kähler manifolds, the Grothendieck K group is isomorphic to the K group generated by vector bundles. So we will write both of these K-groups as $K(X)$;

(ii) for X, a compact Kähler manifold, smooth and proper over Y, $K(X)$ is generated by f-acyclic vector bundles. Indeed, for any coherent sheaf F on X, there

exists an vector bundle resolution $0 \to F \to E_0 \to E_2 \to \cdots \to E_m \to 0$ such that all E_i's are f-acyclic;

(iii) any two f-acyclic vector bundle resolutions of a fixed coherent sheaf in (ii) are dominanted by a common third one.

Now, we are ready to define a morphism $\mathrm{Err}(\cdot; f, P) : K(X) \to H^*(Y, \mathbb{R})$ as follows: For any coherent sheaf F, let $0 \to F \to E_0 \to E_1 \to \cdots \to E_m \to 0$ be a resolution of F by f-acyclic vecor bundles. Set

$$\mathrm{Err}(F; f; P) := \sum_{i=0}^{m} (-1)^i \mathrm{Err}(E_i; f; P).$$

Err is well-defined, i.e., it does not depend on the f-acyclic resolutions of vector bundles we choose. Indeed, if F is a vector bundle, then by Proposition A.2 and (ii) and (iii), we see Err is well-defined. All this, together with (i), implies that the same is true for coherent sheaves, which then completes the proof of Proposition A.3.

(B.4) *Proof of Proposition A.4.* It is clear that we only need to show this for f-acyclic vector bundles. But then

$$\mathrm{Err}(E, \rho; f, \tau_f; P) = \mathrm{ch}_{\mathrm{BC}}(E, \rho; f, \tau_f) - \mathrm{ch}'_{\mathrm{BC}}(E, \rho; f, \tau_f) + f_* \Big(\mathrm{ch}(E) \cdot \mathrm{td}(T_f) \cdot P(T_f) \Big).$$

Now, by Axiom 3 for relative Bott-Chern secondary characteristic classes in 2.C.2, we see that $\mathrm{ch}_{\mathrm{BC}}(E, \rho; f, \tau_f)$ and $\mathrm{ch}'_{\mathrm{BC}}(E, \rho; f, \tau_f)$ are compactible with base change. Obviously, so does the term $f_* \Big(\mathrm{ch}(E) \cdot \mathrm{td}(T_f) \cdot P(T_f) \Big)$. This then proves Proposition A.4.

Remark 5.3. In the past, the functorial rule for relative Bott-Chern secondary characteristic classes are systematically denied by a group of very ill-motivated mathematicians, who simply claim that the functorial rule I used is completely wrong. Unfortunately, they stand on the wrong side. Later we will give more consequences for the functorial rule to indicate its absolute importance.

(B.5) *Proof of Proposition A.5.* Obviously we only need to show it for f-acyclic vector bundles E on X. Choose hermitian metrics ρ_E and ρ_F on E and F respectively, and a hermitian metric τ_f on T_f such that all the induced metrics on the

fibers of f are Kähler. Then, by Propostion A.1,

$$\begin{aligned}
&\mathrm{Err}(E\otimes f^*F;f;P)\\
=&\mathrm{Err}(E\otimes f^*F;\rho\otimes f^*F;f,\tau_f;P)\\
=&\mathrm{ch}_{\mathrm{BC}}(E\otimes f^*F;\rho\otimes f^*F;f,\tau_f)-\mathrm{ch}'_{\mathrm{BC}}(E\otimes f^*F;\rho\otimes f^*F;f,\tau_f)\\
&\qquad +f_*\Big(\mathrm{ch}(E\otimes f^*F)\cdot \mathrm{td}(T_f)\cdot P(T_f)\Big)\quad \text{(by definition)}\\
=&\Big(\mathrm{ch}_{\mathrm{BC}}(E;\rho;f,\tau_f)-\mathrm{ch}'_{\mathrm{BC}}(E;\rho;f,\tau_f)\Big)\cdot \mathrm{ch}(F,\rho_F)\\
&\qquad +f_*\Big(\mathrm{ch}(E)\cdot f^*\mathrm{ch}(F)\cdot \mathrm{td}(T_f)\cdot P(T_f)\Big)\quad \text{(by Axiom 2 in 2.B.2)}\\
=&\mathrm{Err}(E;f;0)\cdot \mathrm{ch}(F,\rho_F)+f_*\Big(\mathrm{ch}(E)\cdot \mathrm{td}(T_f)\cdot P(T_f)\Big)\cdot \mathrm{ch}(F)\\
&\qquad \text{(by definition and the projection formula)}\\
=&\mathrm{Err}(E;f;P)\cdot \mathrm{ch}(F).
\end{aligned}$$

This completes the proof of the proposition.

(B.6) *Proof of Proposition A.6.* In the proof of this proposition, all axioms but the last one, i.e., the rule of the deformation to the normal cone, will be used.

First, it is well-known that, as a $K(Y)$-module, $K(X)$ is generated by line bundles $\mathcal{O}_X$ and $\mathcal{O}_X(-1)$. Obviously, $\mathcal{O}_X$, and $\mathcal{O}_X(-1)$ are both f-acyclic. So by Propositions A.3 and A.5, it is sufficient to show that there exists a unique additive characteristic class R such that

$$\mathrm{Err}(\mathcal{O}_X;f;R)=0,\qquad \text{and}\qquad \mathrm{Err}(\mathcal{O}_X(-1);f;R)=0.$$

We prove this by a direct calculation and Proposition A.4, i.e., the functorial rule.

For the time being, we will assume that F is generated by global sections. Such a condition can be automatically granted if we tensor F by a very ample line bundle on Y. Moreover, it is well-known that such a change will not change the structure f, nor the sheaves $\mathcal{O}$ and $\mathcal{O}(-1)$.

However, with this latest condition, then, we may find a certain natural classifying map $\phi:Y\to G(2,2+n)$, where $G(2,2+n)$ denotes the Grassmannian of dimension two subspaces in a complex $(2+n)$-dimensional space. Moreover, F can be reconstructed as the pull-back of the canonical rank two subbundle S_n of the rank $(2+n)$ trivial bundle on $G(2,2+n)$. By the functorial rule, or better, Proposition A.4, we can then without loss of generality work on the canonical $\mathbb{P}^1$ bundle $p_n:\mathbb{P}(S_n)\to G(2,2+n)$.

Note that in a natural way, $\{p_n\}_{n\geq 0}$ forms an inverse limit system: We indeed has the Cartesian product

$$\begin{array}{ccc} \mathbb{P}(S_n) & \overset{i_{n,m}}{\hookrightarrow} & \mathbb{P}(S_{n+m}) \\ p_n \downarrow & & \downarrow p_{n+m} \\ G(2,2+n) & \overset{j_n}{\hookrightarrow} & G(2,2+(n+m)) \end{array}$$

such that $i_{n,m}^*\mathcal{O}_{\mathbb{P}(S_{n+m})}(k)=\mathcal{O}_{\mathbb{P}(S_n)}(k)$. Hence, by the functorial rule, i.e., Proposition A.5, we see that Err for each p_n for $\mathcal{O}_{\mathbb{P}(S_n)}$ or $\mathcal{O}_{\mathbb{P}(S_n)}(-1)$ forms an inverse limit system in $\{H^*(G(2,2+n),\mathbb{R})\}_{n\geq 0}$.

But it is well-known that

$$H^*(G(2,2+n),\mathbb{R})=\mathbb{R}[c_1(S_n),c_2(S_n),c_1(Q_n),\ldots,c_n(Q_n)]/(c(S_n)\cdot c(Q_n)=1)$$

where Q_n is the canonical quotient bundle on $G(2,2+n)$ which may be defined via the canonical exact sequence

$$0\to S_n\to G(2,2+n)\times\mathbb{C}^{2+n}\to Q_n\to 0,$$

$c(S_n)=1+c_1(S_n)+c_2(S_n)$ and $c(Q_n)=1+c_1(Q_n)+c_2(Q_n)+\cdots+c_n(Q_n)$. (See e.g., [BT, §23].) Note that the relation above says that $c_j(Q_n)$ can be written as a polynomial of $c_1(S_n)$ and $c_2(S_n)$ for $j=1,\ldots,n$. More precisely, for $k=2,\ldots,n$, we have the recurrence formula $c_k(Q_n)=-c_1(S_n)\cdot c_{k-1}(Q_n)-c_2(S_n)\cdot c_{k-2}(Q_n)$ with $c_0(Q_n)=1$, $c_1(Q_n)=-c_1(S_n)$. Hence, in particular, $\lim_{\leftarrow} H^*(G,\mathbb{R})=\mathbb{R}[[c_1,c_2]]$. So, Err for $\mathcal{O}$ and $\mathcal{O}(-1)$ can be written as some universal classes in a ring of power series of c_1 and c_2. Set $Err(\mathcal{O};\pi;P):=P_1(c_1,c_2)$ and $Err(\mathcal{O}(-1);\pi;P):=P_2(c_1,c_2)$ be the two power series.

Moreover, it is clear that if we change S_n by tensoring an ample line bundle L on S_n, $\mathcal{O},\pi$ and P will not be changed. But c_1 and c_2 are changed to $c_1+2c_1(L)$ and $c_2+c_1(L)(c_1+c_1(L))$ respectively. This implies that

$$c_2(S_n)-\frac{1}{4}c_1^2(S_n)=c_2(S_n\otimes L)-\frac{1}{4}c_1^2(S_n\otimes L)$$

remains the same. From this, we see that $P_1(c_1,c_2)$ is acturally a power series in $c_2-\frac{1}{4}c_1^2$, which we denote by P_1 as well by an abuse of notation.

On the other hand, we have the exact sequence

$$0\to\mathcal{O}_X\to p^*S^\vee\otimes\mathcal{O}_X(1)\to T_p\to 0.$$

Hence, $T_p \simeq p^*(\det S)^\vee \otimes \mathcal{O}_X(2)$. So $\mathcal{O}(-1) \otimes \frac{1}{2}p^*(\det S) = \frac{1}{2}T_p^\vee$. But if we tensor S by a line bundle, T_p, associated to p which remains the same, will not be changed. Thus, by Proposition A.5, we see that if we multiply the Err by $\exp(\frac{1}{2}c_1)$, which takes care of the part of $\frac{1}{2}p^*(\det S)$ in $\frac{1}{2}T_p^\vee = \mathcal{O}(-1) \otimes \frac{1}{2}p^*(\det S)$, $P_2(c_1, c_2) = \mathrm{Err}(\mathcal{O}(-1); p; R)$ becomes a power series in $c_2 - \frac{1}{4}c_1^2$ as well, which we still denote by P_2 by an abuse of notation.

With all this, now we are ready to determine Err precisely. For this purpose, first, let $A = c_1(\mathcal{O}(1)) - \frac{1}{2}p^*c_1(S)$. Then, we have

$$c_1(T_p, \tau_p) = 2A, \quad \text{and} \quad A^2 = c_1(\mathcal{O}(1))^2 - c_1(\mathcal{O}(1)) \cdot p^*c_1(S) = p^*(-c_2(F) + \frac{1}{4}c_1(S)^2).$$

Therefore, by the fact that the direct image of the structure sheaf is the structure sheaf on the base, $p_*A = 1$,

$$p_*A^{2m} = 0, \quad \text{and} \quad p_*(A^{2m+1}) = (-c_2(F) + \frac{1}{4}c_1(F)^2)^m p_*A = (-c_2(F) + \frac{1}{4}c_1(F)^2)^m \tag{5.1}$$

for all positive integer m.

Now, we want to see how Err depends on P. Obviously, if we change P by adding P', then the corresponding Err for $\mathcal{O}_X$ changes by

$$p_*(\frac{2A}{1 - e^{-2A}} P'(2A))$$

and similarly, for $\mathcal{O}_X(-1)$, (up to the factor $\exp(\frac{1}{2}c_1)$ as discussed above,) the error changes by

$$p_*(\frac{2Ae^{-A}}{1 - e^{-2A}} P'(2A)).$$

In particular, for $\mathcal{O}(-1)$, the factor before P' is an even function in A, and hence is a series in even powers. So, by (5.1), we may choose a unique odd power series R^{odd}, such that for any even power series P',

$$\mathrm{Err}(\mathcal{O}_X(-1); p; R^{\mathrm{odd}} + P') = Err(\mathcal{O}_X(-1); p; R^{\mathrm{odd}}) = 0.$$

Similarly, as for $\mathcal{O}$, once the odd part of P is fixed, by (5.1) again, there is one and only one even power series R^{even} such that

$$\mathrm{Err}(\mathcal{O}_X; p; R^{\mathrm{odd}} + R^{\mathrm{even}}) = 0,$$

as now only the odd terms of $\frac{2A}{1-e^{-2A}}(R^{\mathrm{odd}}(2A) + P'(2A))$ matters.

Therefore, finally, we see that there exists a unique power series $R = R^{\text{even}} + R^{\text{odd}}$ such that $\text{Err}(\cdot; p; R) = 0$ for all $\mathbb{P}^1$-bundles. This completes the proof of Proposition A.6.

(B.7) *Proof of Proposition A.7.* We divide the proof of this proposition into two steps;

(i) Consider the special situation where codimension-one closed immersions are sections of $\mathbb{P}^1$-bundles. In this case, we have the following

Lemma 1. *Let $f : X \to Y$ be a smooth, proper morphism of compact Kähler manifolds, and $\pi : \mathbb{P}^1_X \to X$ be a $\mathbb{P}^1$-bundle over X. Assume that $i : X \to \mathbb{P}^1_X$ is a section, so that $\pi \circ i = Id_X$. Then for the closed immersion i over Y, and any vector bundle E on X, we have*

$$Err(E; i; P) = 0.$$

(ii) Deduce a general codimension-one closed immersion to a section of a $\mathbb{P}^1$-bundle by the deformation to the normal cone technique. More precise, we have the following

Lemma 2. *With the same notation as in 3.A.1, i.e., suppose we have the diagram*

$$\begin{array}{ccccc} \mathbb{P}^1_X & \overset{i_\infty}{\hookleftarrow} & X & \overset{i_0=i}{\hookrightarrow} & Z \\ & g_\infty \searrow & f_\infty \downarrow f_0 & \swarrow g_0 & \\ & & Y, & & \end{array}$$

then for any vector bundle E on Z, we have

$$Err\Big((i_0)_*(i^*E); g_0; P\Big) = Err\Big((i_\infty)_*(i^*E); g_\infty; P\Big).$$

Before proving these two lemmas, let us show how Proposition A.7 follows from them.

First of all, by Lemma 2, we see that for all coherent sheaves F on Z,

$$\text{Err}((i_0)_* i^* F; g_0; P) = \text{Err}((i_\infty)_* i^* F; g_\infty; P).$$

Obviously, all coherent sheaves on X can be realized as the pull-back of coherent sheaves from Z, so, for all coherent sheaves F' on X,

$$\text{Err}((i_0)_* F'; g_0; P) = \text{Err}((i_\infty)_* F'; g_\infty; P).$$

This then certainly implies that for all vector bundles E' on X,

$$\mathrm{Err}((i_0)_*E'; g_0; P) = \mathrm{Err}((i_\infty)_*E'; g_\infty; P). \tag{5.2}$$

On the other hand, by definition, $f_0 = f_\infty$, we have

$$\mathrm{Err}(E'; f_0; P) = \mathrm{Err}(E'; f_\infty; P). \tag{5.3}$$

Thererfore,

$$\begin{aligned}
&\mathrm{Err}(E'; i_0; P)\\
=&\mathrm{Err}(E'; f_0; P) - \mathrm{Err}((i_0)_*E'; g_0; P) \qquad \text{(by definition)}\\
=&\mathrm{Err}(E'; f_\infty; P) - \mathrm{Err}((i_0)_*E'; g_0; P) \qquad \text{(by (5.3))}\\
=&\mathrm{Err}(E'; f_\infty; P) - \mathrm{Err}((i_\infty)_*E'; g_\infty; P) \qquad \text{(by (5.2))}\\
=&\mathrm{Err}(E'; i_\infty; P) \qquad \text{(by definition)}\\
=&0 \quad \text{(by Lemma 1).}
\end{aligned}$$

This then is exactly what we want to show in Proposition A.7.

Now we go back to show Lemma 1 and Lemma 2.

(i) *Proof of Lemma 1.* Here that i is a section of a $\mathbb{P}^1$-bundle over X plays a very important rule. Indeed, if g denotes the natural composition $\mathbb{P}^1_X \xrightarrow{\pi} X \xrightarrow{f} Y$, then by definition,

$$\begin{aligned}
&\mathrm{Err}(E; i; P)\\
=&\mathrm{Err}(E; f; P) - \mathrm{Err}(i_*E; g; P)\\
=&\mathrm{Err}(E; f; P) - \mathrm{Err}(i_*E; f \circ \pi; P)\\
=&\mathrm{Err}(E; f; P) - \Big(\mathrm{Err}(\pi_* \circ i_*E; f; P) + f_*\big(\mathrm{Err}(i_*E; \pi; P) \cdot \mathrm{td}(T_f)\big)\Big)\\
&\qquad \text{(by Proposition A.2)}\\
=&\mathrm{Err}(E; f; P) - \Big(\mathrm{Err}((\mathrm{Id}_X)_*E; f; P) + f_*\big(\mathrm{Err}(i_*E; \pi; R) \cdot \mathrm{td}(T_f)\big)\Big)\\
&\qquad \text{(as } \pi \circ i = \mathrm{id}_X \text{ and by our assumption on } P)\\
=&-f_*\big(\mathrm{Err}(i_*E; \pi; R) \cdot \mathrm{td}(T_f)\big)\\
=&-f_*\big(0 \cdot \mathrm{td}(T_f)\big) \quad \text{(by Proposition A.6)}\\
=&0.
\end{aligned}$$

Remark 5.4. Note that in this proof, we only use the fact that i is a section of a $\mathbb{P}^1$-bundle and for $\mathbb{P}^1$-bundles, Err is simply zero. Thus if i is a section of a $\mathbb{P}^n$-bundle

and, for all $\mathbb{P}^n$-bundles, Err is zero as well, we may have a similar discussion. We will use this remark in the next chapter.

(ii) *Proof of Lemma 2.* This is where we should use the deformation to the normal cone rule. So we recall the following commutative diagram.

$$
\begin{array}{ccccccccc}
X \times \{t\} & & \hookrightarrow & & X \times \mathbb{P}^1 & & \hookleftarrow & & X \times \{\infty\} \\
\Big| & i_t \searrow & & & I \downarrow & & & \swarrow i_\infty & \Big| \\
\Big| & (t \neq \infty) & Z \times \{t\} & \hookrightarrow & B_{X \times \{\infty\}} Z \times \mathbb{P}^1 & \hookleftarrow & \begin{array}{c}\mathbb{P}\\ + \\ Z\end{array} & \searrow g_\infty & \Big| \\
f_t & & & \searrow & \pi \downarrow & \swarrow & & & f_\infty \\
\Big| & g_t \swarrow & & & Z \times \mathbb{P}^1 & & & \searrow & \Big| \\
\downarrow & & & & \downarrow & & & & \downarrow \\
Y \times \{t\} & & \hookrightarrow & & Y \times \mathbb{P}^1 & & \hookleftarrow & & Y \times \{\infty\}
\end{array}
$$

By pulling back E from Z onto $W = B_{X \times \{\infty\}} Z \times \mathbb{P}^1$ via the composition of natural maps $W \xrightarrow{\pi} Z \times \mathbb{P}^1 \to Z$, we obtain a vector bundle on W, denoted by E as well by an abuse of notation. Twisted by $B_X Z$, we then get the exact sequence of coherent sheaves on W;

$$
0 \to E(B_X Z - X \times \mathbb{P}^1) \to E(B_X Z) \to I_* I^* \Big(E(B_X Z) \Big) \to 0. \tag{5.4}
$$

Note that here in particular $E(B_X Z - X \times \mathbb{P}^1)$ and $E(B_X Z)$ are vector bundles on W. Easily, we have

$$
\begin{aligned}
I_* I^* \Big(E(B_X Z) \Big) \Big|_{X \times \{t\}} &= (i_t)_* (i^* E), \quad \text{if } t \neq \infty; \\
I_* I^* \Big(E(B_X Z) \Big) \Big|_{\mathbb{P}} &= (i_\infty)_* (i^* E),
\end{aligned}
$$

and

$$
E(B_X Z - X \times \mathbb{P}^1) = \pi^* (E(-X \times \mathbb{P}^1) \otimes \mathcal{O}_{Z \times \mathbb{P}^1}(Z \times \{\infty\})) = \pi^* (p_Z^* (E(-X)) \otimes p_{\mathbb{P}^1}^* (\mathcal{O}_{\mathbb{P}^1}(\infty)).
$$

Here p_Z and $p_{\mathbb{P}^1}$ denote the projection of $Z \times \mathbb{P}^1$ to Z and $\mathbb{P}^1$ respectively.

On the other hand, by the fact that $W \to \mathbb{P}^1$ is flat, the restrictions of (5.4) to $Z \times \{t\}$ and $\mathbb{P} \cup B_X Z$ are again exact. In particular, with the same notation as in 3.A.2, we have the short exact sequences

$$
0 \to E_t(-X) \to E_t \to (i_t)_* i_t^* E_t \to 0 \qquad \text{on} \quad Z \times \{t\};
$$

$$0 \to E_\infty(-X) \to E_\infty \to (i_\infty)_* i_\infty^* E_\infty \to 0 \qquad \text{on} \quad \mathbb{P};$$

and

$$0 \to E'_\infty \to E''_\infty \to 0 \to 0 \qquad \text{on} \quad B_X Z.$$

Thus, by Proposition A.3, it is sufficient to show that

$$\begin{aligned}
&\mathrm{Err}\Big(E(B_X Z - X \times \mathbb{P}^1)\Big|_{Z\times\{0\}}; g_0; P\Big) - \mathrm{Err}\Big(E(B_X Z)\Big|_{Z\times\{0\}}; g_0; P\Big)\\
=&\mathrm{Err}\Big(E(B_X Z - X \times \mathbb{P}^1)\Big|_{\mathbb{P}}; g_\infty; P\Big) - \mathrm{Err}\Big(E(B_X Z)\Big|_{\mathbb{P}}; g_\infty; P\Big),
\end{aligned}$$

or better, to show that

$$\mathrm{Err}\Big(E(-X); g_0; P\Big) - \mathrm{Err}\Big(E; g_0; P\Big) = \mathrm{Err}\Big(E_\infty(-X); g_\infty; P\Big) - \mathrm{Err}\Big(E_\infty; g_\infty; P\Big).$$

Here E_∞ denotes $E(B_X Z)\Big|_{\mathbb{P}}$. From here, by Proposition A.3 again, without loss of generality, we may assume that E_t and $E_t(-X)$ are all g_t-acyclic.

Lemma 3. *With the same notation as above, for all $t \neq \infty$,*

$$Err\Big(E(-X); g_0; P\Big) - Err\Big(E; g_0; P\Big) = Err\Big(E_t(-X); g_t; P\Big) - Err\Big(E_t; g_t; P\Big).$$

Proof. We first show that

$$\mathrm{Err}\Big(E; g_0; P\Big) = \mathrm{Err}\Big(E_t; g_t; P\Big). \tag{5.5}$$

Choose a metric ρ on E as a vector bundle on Z. By Proposition A.1,

$$\begin{aligned}
&\mathrm{Err}\Big(E; g_0; P\Big)\\
=&\mathrm{ch}_{\mathrm{BC}}(E_0, \rho; g_0, \tau_{g_0}) - \mathrm{ch}'_{\mathrm{BC}}(E_0, \rho; g_0, \tau_{g_0}) + (g_0)_*\Big(\mathrm{ch}(E_0) \cdot \mathrm{td}(T_{g_0}) \cdot P(T_{g_0})\Big),
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Err}\Big(E_t; g_t; P\Big)\\
=&\mathrm{ch}_{\mathrm{BC}}(E_t, \rho_t; g_t, \tau_{g_t}) - \mathrm{ch}'_{\mathrm{BC}}(E_t, \rho_t; g_t, \tau_{g_t}) + (g_t)_*\Big(\mathrm{ch}(E_t) \cdot \mathrm{td}(T_{g_t}) \cdot P(T_{g_t})\Big).
\end{aligned}$$

Note that in the construction of the deformation to the normal cone, the base curve is a rational curve $\mathbb{P}^1$. As a direct consequence, the de Rham cohomology classes will not be changed for this deformation family. This then shows that in $H^*(Y, \mathbb{R})$, for all $t \neq \infty$,

$$(g_t)_*\Big(\mathrm{ch}(E_t) \cdot \mathrm{td}(T_{g_t}) \cdot P(T_{g_t})\Big) = (g_0)_*\Big(\mathrm{ch}(E_0) \cdot \mathrm{td}(T_{g_0}) \cdot P(T_{g_0})\Big). \tag{5.6}$$

Note that $E_0 \simeq E_t$ and $T_{g_0} \simeq T_{g_t}$, so by changing from 0 to t, we may understand that we are working on the same fibration $g : Z \to Y$ for the same g-acyclic vector bundle E. Hence, from $(E_0, \rho_0; g_0, \tau_0)$ to $(E_t, \rho_t; g_t, \tau_t)$, the only changes come from the metrics: for E, the metric ρ is changed to ρ_t while for T_g, the metric τ_{g_0} is changed to τ_{g_t}. Thus by Axiom 4 and Axiom 5 for relative Bott-Chern secondary characteristic classes in 2.D.2 and 2.E.3 respectively, we see that the change of relative Bott-Chern classes from 0 to t for $\mathrm{ch}_{\mathrm{BC}}$ is the same as the change of relative Bott-Chern classes from 0 to t for $\mathrm{ch}'_{\mathrm{BC}}$, as described in 3.B.2. More precisely, by (3.3.7),

$$\begin{aligned}
&\mathrm{ch}_{\mathrm{BC}}(E_0, \rho; g_0, \tau_{g_0}) - \mathrm{ch}_{\mathrm{BC}}(E_t, \rho_t; g_t, \tau_{g_t}) \\
=& - g_* \Big(\mathrm{ch}_{\mathrm{BC}}(E; \rho_t, \rho_0) \cdot \mathrm{td}(T_g, \tau_t) + \mathrm{ch}(E; \rho_0) \cdot \mathrm{td}_{\mathrm{BC}}(T_g; \tau_t, \tau_0) \Big) \\
&\qquad - \mathrm{ch}_{\mathrm{BC}} \Big(g_* E; L^2(\rho_t, \tau_t), L^2(\rho_0, \tau_0) \Big) \\
=& \mathrm{ch}'_{\mathrm{BC}}(E_0, \rho; g_0, \tau_{g_0}) - \mathrm{ch}'_{\mathrm{BC}}(E_t, \rho; g_t, \tau_{g_t}).
\end{aligned}$$

This, together with (5.6), then certainly gives (5.5). Similarly, we have

$$\mathrm{Err}\Big(E(-X); g_0; P\Big) = \mathrm{Err}\Big(E_t(-X); g_t; P\Big). \tag{5.7}$$

So finally, by (5.5) and (5.7), we complete the proof of Lemma 3.

Now we are finally ready to complete the proof of Proposition A.7.

For $t \neq \infty$ in the projective line $\mathbb{P}^1$, we have

$$
\begin{aligned}
&\Big(\mathrm{Err}(E_0(-X); g_0; P) - \mathrm{Err}(E_0; g_0; P)\Big) \\
&\qquad - \Big(\mathrm{Err}(E_\infty(-X); g_\infty; P) - \mathrm{Err}(E_\infty; g_\infty; P)\Big) \\
=&\Big(\mathrm{Err}(E_t(-X); g_t; P) - \mathrm{Err}(E_t; g_t; P)\Big) \\
&\qquad - \Big(\mathrm{Err}(E_\infty(-X); g_\infty; P) - \mathrm{Err}(E_\infty; g_\infty; P)\Big) \\
&\qquad \text{(by Lemma 3)} \\
=&\Big(\mathrm{ch}_{\mathrm{BC}}(E_t(-X), \rho'_t; g_t, \tau_{g_t}) - \mathrm{ch}'_{\mathrm{BC}}(E_t(-X), \rho'_t; g_t, \tau_{g_t}) \\
&\qquad + (g_t)_*\big(\mathrm{ch}(E_t(-X)) \cdot \mathrm{td}(T_{g_t}) \cdot P(T_{g_t})\big)\Big) \\
&- \Big(\mathrm{ch}_{\mathrm{BC}}(E_t, \rho_t; g_t, \tau_{g_t}) - \mathrm{ch}'_{\mathrm{BC}}(E_t, \rho_t; g_t, \tau_{g_t}) \\
&\qquad + (g_t)_*\big(\mathrm{ch}(E_t) \cdot \mathrm{td}(T_{g_t}) \cdot P(T_{g_t})\big)\Big) \\
&- \Big(\mathrm{ch}_{\mathrm{BC}}(E_\infty(-X), \rho'_\infty; g_\infty, \tau_{g_\infty}) - \mathrm{ch}'_{\mathrm{BC}}(E_\infty(-X), \rho'_\infty; g_\infty, \tau_{g_\infty}) \\
&\qquad + (g_\infty)_*\big(\mathrm{ch}(E_\infty(-X)) \cdot \mathrm{td}(T_{g_\infty}) \cdot P(T_{g_\infty})\big)\Big) \\
&+ \Big(\mathrm{ch}_{\mathrm{BC}}(E_\infty, \rho_\infty; g_\infty, \tau_{g_\infty}) - \mathrm{ch}'_{\mathrm{BC}}(E_\infty, \rho_\infty; g_\infty, \tau_{g_\infty}) \\
&\qquad + (g_\infty)_*\big(\mathrm{ch}(E_\infty) \cdot \mathrm{td}(T_{g_\infty}) \cdot P(T_{g_\infty})\big)\Big) \\
&\qquad \text{(by definition of Err)} \\
=&\Big(\Big(\mathrm{ch}_{\mathrm{BC}}(E_t(-X), \rho'_t; g_t, \tau_{g_t}) - \mathrm{ch}_{\mathrm{BC}}(E_t, \rho_t; g_t, \tau_{g_t})\Big) \\
&\qquad - \Big(\mathrm{ch}'_{\mathrm{BC}}(E_t(-X), \rho'_t; g_t, \tau_{g_t}) - \mathrm{ch}'_{\mathrm{BC}}(E_t, \rho_t; g_t, \tau_{g_t})\Big)\Big) \\
&- \Big(\Big(\mathrm{ch}_{\mathrm{BC}}(E_\infty(-X), \rho'_\infty; g_\infty, \tau_{g_\infty}) - \mathrm{ch}_{\mathrm{BC}}(E_\infty, \rho_\infty; g_\infty, \tau_{g_\infty})\Big) \\
&\qquad - \Big(\mathrm{ch}'_{\mathrm{BC}}(E_\infty(-X), \rho'_\infty; g_\infty, \tau_{g_\infty}) - \mathrm{ch}'_{\mathrm{BC}}(E_\infty, \rho_\infty; g_\infty, \tau_{g_\infty})\Big)\Big) \\
&+ \Big((g_t)_*\big(\mathrm{ch}(E_t(-X) - E_t) \cdot \mathrm{td}(T_{g_t}) \cdot P(T_{g_t})\big) \\
&\qquad - (g_\infty)_*\big(\mathrm{ch}(E_\infty(-X) - E_\infty) \cdot \mathrm{td}(T_{g_\infty}) \cdot P(T_{g_\infty})\big)\Big) \\
=&\Big(\Big(\mathrm{ch}_{\mathrm{BC}}(E_t(-X), \rho'_t; g_t, \tau_{g_t}) - \mathrm{ch}_{\mathrm{BC}}(E_t, \rho_t; g_t, \tau_{g_t})\Big) \\
&\qquad - \Big(\mathrm{ch}'_{\mathrm{BC}}(E_t(-X), \rho'_t; g_t, \tau_{g_t}) - \mathrm{ch}'_{\mathrm{BC}}(E_t, \rho_t; g_t, \tau_{g_t})\Big)\Big) \\
&- \Big(\Big(\mathrm{ch}_{\mathrm{BC}}(E_\infty(-X), \rho'_\infty; g_\infty, \tau_{g_\infty}) - \mathrm{ch}_{\mathrm{BC}}(E_\infty, \rho_\infty; g_\infty, \tau_{g_\infty})\Big) \\
&\qquad - \Big(\mathrm{ch}'_{\mathrm{BC}}(E_\infty(-X), \rho'_\infty; g_\infty, \tau_{g_\infty}) - \mathrm{ch}'_{\mathrm{BC}}(E_\infty, \rho_\infty; g_\infty, \tau_{g_\infty})\Big)\Big) \\
&+ \Big((g_0)_*\big(\mathrm{ch}(E_0(-X) - E_0) \cdot \mathrm{td}(T_{g_0}) \cdot R(T_{g_0})\big) \\
&\qquad - (g_\infty)_*\big(\mathrm{ch}(E_\infty(-X) - E_\infty) \cdot \mathrm{td}(T_{g_\infty}) \cdot P(T_{g_\infty})\big)\Big) \\
&\qquad \text{(by (5.6)).}
\end{aligned}
$$

From here, if we use Axiom 6 of the deformation to the normal cone rule for the relative Bott-Chern secondary characteristic classes in 3.C.3, by taking the limit $t \to \infty$, we see that in the above expression, the contribution of $\mathrm{ch}_{\mathrm{BC}}$ exactly cancels out the contribution from $\mathrm{ch}'_{\mathrm{BC}}$. This then implies that

$$\begin{aligned}
&\Big(\mathrm{Err}(E_0(-X); g_0; P) - \mathrm{Err}(E_0; g_0; P)\Big) \\
&\qquad - \Big(\mathrm{Err}(E_\infty(-X); g_\infty; P) - \mathrm{Err}(E_\infty; g_\infty; P)\Big) \\
=&(g_0)_*\big(\mathrm{ch}(E_0(-X) - E_0) \cdot \mathrm{td}(T_{g_0}) \cdot R(T_{g_0})\big) \\
&\qquad - (g_\infty)_*\big(\mathrm{ch}(E_\infty(-X) - E_\infty) \cdot \mathrm{td}(T_{g_\infty}) \cdot P(T_{g_\infty})\big).
\end{aligned} \tag{5.8}$$

Lemma 4. *With the same notation as above,*

$$(g_0)_*\Big(ch(E_0(-X)-E_0)\cdot td(T_{g_0})\cdot R(T_{g_0})\Big) = (g_\infty)_*\Big(ch(E_\infty(-X)-E_\infty)\cdot td(T_{g_\infty})\cdot P(T_{g_\infty})\Big).$$

Proof. Let N_{i_t} denotes the normal bundle of i_t for all $t \in \mathbb{P}^1$. Then we have the following exact sequence

$$0 \to T_f \to i_t^* T_{g_t} \to N_{i_t} \to 0.$$

Therefore, $i_t^*\mathrm{td}(T_{g_t}) = \mathrm{td}(T_f) \cdot \mathrm{td}(N_{i_t})$ and $i_t^* P(T_{g_t}) = P(T_f) + P(N_{i_t})$. Hence, for all points t in the projective line $\mathbb{P}^1$,

$$\begin{aligned}
&(g_t)_*\Big(\mathrm{ch}(E_t(-X) - E_t) \cdot \mathrm{td}(T_{g_t}) \cdot P(T_{g_t})\Big) \\
=&(g_t)_*\Big(\mathrm{ch}((i_t)_* i_t^* E) \cdot \mathrm{td}(T_{g_t}) \cdot P(T_{g_t})\Big) \\
=&(g_t)_*\Big((i_t)_*\big(\mathrm{ch}(i_t^* E) \cdot \mathrm{td}(N_{i_t})^{-1}\big) \cdot \mathrm{td}(T_{g_t}) \cdot P(T_{g_t})\Big) \\
&\qquad \text{(by Grothendieck-Riemann-Roch Theorem in Algebraic Geometry for } i_t) \\
=&(g_t)_*\Big((i_t)_*\big(\mathrm{ch}(i_t^* E) \cdot \mathrm{td}(N_{i_t})^{-1} \cdot i_t^*\mathrm{td}(T_{g_t}) \cdot i_t^* P(T_{g_t})\big)\Big) \\
&\qquad \text{(by the projection formula)} \\
=&(g_t)_*\Big((i_t)_*\big(\mathrm{ch}(i_t^* E) \cdot \mathrm{td}(N_{i_t})^{-1} \cdot \mathrm{td}(T_f) \cdot \mathrm{td}(N_{i_t}) \cdot (P(T_f) + P(N_{i_t}))\big)\Big) \\
=&f_*\Big(\mathrm{ch}(i_t^* E) \cdot \mathrm{td}(N_{i_t})^{-1} \cdot \mathrm{td}(T_f) \cdot \mathrm{td}(N_{i_t}) \cdot (P(T_f) + P(N_{i_t}))\Big) \\
=&f_*\Big(\mathrm{ch}(i^* E) \cdot \mathrm{td}(T_f) \cdot (P(T_f) + P(N_{i_t}))\Big).
\end{aligned}$$

On the other hand, $P(N_{i_\infty}) = P(N_{i_t})$. Indeed, since $X \times \mathbb{P}^1$ does not meet $B_X Z$, and $W \to \mathbb{P}^1$ is flat, the restriction of the normal bundle N_I of the closed immersion $I : X \times \mathbb{P}^1$ to $X \times \{t\}$ is simply N_t for all $t \in \mathbb{P}^1$. Therefore, the cycles corresponding

to $\mathrm{ch}(N_{i_t})$ and $\mathrm{ch}(N_\infty)$ are the same for all t. So do the corresponding de Rham cohomology classes. This then implies $P(N_{i_\infty}) = P(N_{i_t})$ in $H^*(Y, \mathbb{R})$, and hence completes the proof of Lemma 4.

Thus, we finally have;

$$\begin{aligned}
&\mathrm{Err}(i_* i^* E; g; P) - \mathrm{Err}((i_\infty)_* i_\infty^* E; g_\infty; P) \\
=&\Big(\mathrm{Err}(E_0(-X); g_0; P) - \mathrm{Err}(E_0; g_0; P)\Big) \\
&\qquad - \Big(\mathrm{Err}(E_\infty(-X); g_\infty; P) - \mathrm{Err}(E_\infty; g_\infty; P)\Big) \quad \text{(by definition)} \\
=&(g_0)_*\big(\mathrm{ch}(E_0(-X) - E_0) \cdot \mathrm{td}(T_{g_0}) \cdot R(T_{g_0})\big) \\
&\qquad - (g_\infty)_*\big(\mathrm{ch}(E_\infty(-X) - E_\infty) \cdot \mathrm{td}(T_{g_\infty}) \cdot P(T_{g_\infty})\big) \quad \text{(by (5.8))} \\
=&0 \qquad \text{(by Lemma 4)},
\end{aligned}$$

which completes the proof of Proposition A.7.

6. Proof of the Uniqueness for Relative Bott-Chern Secondary Characteristic Classes

A) Rank n-Projective Bundles

B) Closed Immersions of Higher Codimension

C) Proof of Uniqeness Theorems

In this chapter, we will complete the proof of the uniqueness theorems for relative Bott-Chern secondary characteristic classes. For doing so, we need to study Err in two cases: $\mathbb{P}^n$-bundles and general closed immersions of codimension m. We will use a trick of Bott to deduce Err for a $\mathbb{P}^n$-bundle and use a trick of Faltings to deduce Err for a closed immersion of higher codimension to the cases of $\mathbb{P}^1$-bundles and closed immersions of codimension 1. Once this is achieved, we then can use Propositions 5.A.6 and 5.A.7 to finish the proof.

A. Rank n-Projective Bundles

(A.1) For simplicity, in this section, we will use p_n to denote the projection from any $\mathbb{P}^n$-bundle to its base, and use i_1 to denote any codimension one closed immersion.

In order to prove Err's are zero for projections p_n from $\mathbb{P}^n$-bundles, we use an induction on n. In a more practical term, we deduce the problems for Err with respect to p_n to these for just p_1 and i_1.

If $n = 1$, by Proposition 5.A.6, we know that there exists a characteristic class R for rank two bundles such that

$$\mathrm{Err}(\cdot; p_1; R) \equiv 0. \tag{6.1}$$

Assume now that for any $m < n$, we have a characteristic class R_m for rank m vector bundles such that

(1_m) R_m is additive and $R_m(E) = R_{m-1}(E)$ for all rank $m-1$ vector bundles;

(2_m) $\mathrm{Err}(\cdot; p_m; R_m) \equiv 0$.

We then want to prove that there exists a characteristic class R_n for rank n vector bundles such that

(1_n) R_n is additive and $R_n(E) = R_{n-1}(E)$ for all rank $n-1$ vector bundles;

(2_n) $\mathrm{Err}(\cdot; p_n; R_n) \equiv 0$.

(A.2) In order to prove this, consider the generator of $K(X)$ for $X = \mathbb{P}_Y(F)$, where F is a rank $n+1$ vector bundle on Y. Note that by using a classifying map to Grassmannian, we may assume that F has a rank 1 sub-line bundle L such that F/L is again a vector bundle. In particular, we have the following closed embedding of codimension one:

$$\begin{array}{ccccc} \mathbb{P}_Y(F/L) & & \stackrel{i=i_1}{\hookrightarrow} & & \mathbb{P}_Y(F) \\ & p_{n-1} \searrow & & \swarrow p_n & \\ & & Y. & & \end{array}$$

Lemma 1. *As a $K(Y)$-module, $K(X)$ is generated by $\mathcal{O}_X(-1)$ and the direct image of $i_*\Big(K(\mathbb{P}_Y(F/L))\Big)$.*

Proof. This is a direct consequence of the facts that

(i) we have a short exact sequence

$$0 \to L \to F \to F/L \to 0$$

and hence $\mathrm{ch}(F) = \mathrm{ch}(F/L) + \mathrm{ch}(L)$;

(ii) $K(\mathbb{P}^n_Y)$ is generated by $\mathcal{O}(i)$, $i = 1, \ldots, n$ and $i_*(\mathcal{O}(a)) = \mathcal{O}(a) - \mathcal{O}(a-1)$ from the structure exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to i_*\mathcal{O} \to 0.$$

(A.3) With this lemma, in order to prove the uniqueness theorems, we only need to show that there exists an R_n satisfies (1_n) and (2_n) for $\mathcal{O}_X(-1)$ and all the elements in $i_*\Big(K(\mathbb{P}_Y(F/L))\Big)$.

We first deal with the elements in $i_*\Big(K(\mathbb{P}_Y(F/L))\Big)$. For this purpose, we use Proposition 5.A.7. In fact, since

$$i_1 : \mathbb{P}_Y(F/L) \hookrightarrow \mathbb{P}_Y(F)$$

is a codimension-one closed imbedding, we have $\mathrm{Err}(\alpha; i_1; R_n) = 0$ for all $\alpha \in K(\mathbb{P}_Y(F/L))$. But by definition,

$$\mathrm{Err}(\alpha; i_1; R_n) = \mathrm{Err}(\alpha; p_{n-1}; R_n) - \mathrm{Err}((i_1)_*\alpha; p_n; R_n).$$

Here p_{n-1} (resp. p_n) denotes the natural projection from $\mathbb{P}_Y(F)$ (resp. $\mathbb{P}_Y(F/L)$) to Y. This implies that $\mathrm{Err}(\alpha; p_{n-1}; R_n) = \mathrm{Err}((i_1)_*\alpha; p_n; R_n)$.

Now, by the induction hypothesis (1_m) and (2_m), $m \leq n-1$,

$$\mathrm{Err}(\alpha; p_{n-1}; R_n = R_{n-1}) \equiv 0,$$

hence we have

$$\mathrm{Err}((i_1)_*\alpha; p_n; R_n) \equiv 0,$$

which exactly means that (1_n) and (2_n) are valid for the elements in the direct image of $K(\mathbb{P}_Y(F/L))$.

(A.4) Now let us study $\mathrm{Err}(\mathcal{O}_X(-1), p_n; R_n)$. For this special purpose, we use a trick of Bott following Faltings [F].

Let $\mathrm{Flag}_Y(F)$ be the Flag variety of F on Y. That is, the variety which classifies complete filtrations of F:

$$0 = F_0 \subset F_1 \subset ... \subset F_{n+1} = F,$$

where the successive vector bundle quotients are of rank 1. There is a natural morphism from $\mathrm{Flag}_Y(F)$ to X which is just the composition of the forgetting maps. Hence the morphism from $\mathrm{Flag}_Y(F)$ to X is a composition of $\mathbb{P}^m$-bundles with $m < n$. Therefore, by Proposition 5.A.2, and the induction hypothesis, Err becomes zero for the morphism $\mathrm{Flag}_Y(F) \to X$ and any additive R such that $R = R_m$ for rank $m < n$ vector bundles.

On the other hand, consider the pull-back of the line bundle $\mathcal{O}_X(-1)$ over $\mathrm{Flag}_Y(F)$. By a simple calculation using the projection formula and the fact that the direct image of the structure sheaf on the total space $\mathrm{Flag}_Y(F)$ is simply the structure sheaf on the base X, we see that the push-forward to X of this pull-back line bundle, denoted by $\mathcal{O}'(-1)$, on $\mathrm{Flag}_Y(F)$ coincides with $\mathcal{O}_X(-1)$ itself. Note that $\mathcal{O}'(-1)$ is $(\mathrm{Flag}_Y(F) \to X)$-acyclic and its direct image $\mathcal{O}_X(-1)$ is $(X \to Y)$-acyclic. Thus by Proposition 5.A.2, we have

$$\begin{aligned}
&\mathrm{Err}(\mathcal{O}'(-1); \mathrm{Flag}_Y(F) \to X \overset{p_n}{\to} Y; R)\\
=&(p_n)_*\Big(\mathrm{Err}(\mathcal{O}'(-1); \mathrm{Flag}_Y(F) \to X; R) \cdot \mathrm{td}(T_{p_n})\Big) + \mathrm{Err}(\mathcal{O}_X(-1); p_n; R)\\
=&(p_n)_*\Big(0 \cdot \mathrm{td}(T_{p_n})\Big) + \mathrm{Err}(\mathcal{O}_X(-1); p_n; R) \quad \text{(by induction hypothesis)}\\
=&\mathrm{Err}(\mathcal{O}_X(-1); p_n; R).
\end{aligned} \tag{6.2}$$

Therefore, it is sufficient to show that, for the natural morphism $\mathrm{Flag}_Y(F) \to Y$ that Err for $\mathcal{O}'(-1)$, the pull-back of $\mathcal{O}_X(-1)$, is zero.

In order to deal with the morphism $\mathrm{Flag}_Y(F) \to Y$, we introduce another decomposition: Let $\mathrm{Flag}'_Y(F)$ be the flag variety which classifies the following partial filtrations of F:

$$0 = F_0 \subset F_2 \subset \ldots \subset F_{n+1} = F,$$

where the rank of F_k is k. Then the composition of the natural morphism from $\mathrm{Flag}_Y(F)$ to $\mathrm{Flag}'_Y(F)$ with the natural morphism from $\mathrm{Flag}'_Y(F)$ to Y is just $\mathrm{Flag}_Y(F) \to Y$. But, the morphism from $\mathrm{Flag}_Y(F)$ to $\mathrm{Flag}'_Y(F)$ is a $\mathbb{P}^1$-bundle. Therefore, Err for $\mathcal{O}'(-1)$ on $\mathrm{Flag}_Y(F)$ (with respect to the morphism $\mathrm{Flag}_Y(F) \to \mathrm{Flag}'_X F$) vanishes, by our Proposition 5.A.7 for $\mathbb{P}^1$-bundles. On the other hand, the push-forward of $\mathcal{O}'(-1)$ via the $\mathbb{P}^1$-bundle $\mathrm{Flag}_Y(F) \to \mathrm{Flag}'_X F$ to $\mathrm{Flag}'_X F$ is the zero bundle, which certainly is $(\pi : \mathrm{Flag}'_X F \to Y)$-acyclic and satisfies the trivial condition that $\mathrm{Err}(0; \mathrm{Flag}'_X F \to Y; R) = 0$ by Proposition 5.A.3. Therefore, by Proposition 5.A.2 again, we have

$$\begin{aligned}
&\mathrm{Err}(\mathcal{O}'(-1); \mathrm{Flag}_X F \to Y; R)\\
=&\pi_*\Big(\mathrm{Err}(\mathcal{O}'(-1); \mathrm{Flag}_X F \to \mathrm{Flag}'_X F; R) \cdot \mathrm{td}(T_{\mathrm{Flag}'_X F \to Y})\Big)\\
&\qquad + \mathrm{Err}(0; \mathrm{Flag}_X F \to Y; R)\\
=&\pi_*\Big(0 \cdot \mathrm{td}(T_{\mathrm{Flag}'_X F \to Y})\Big) + 0\\
&\qquad \text{(by the fact that } \mathrm{Flag}_X F \to \mathrm{Flag}'_X F \text{ is a } \mathbb{P}^1 - \text{bundle)}\\
=&0.
\end{aligned}$$

This, by (6.2), then gives $\mathrm{Err}(\mathcal{O}_X(-1); X \to Y; R) = 0$, and hence completes the proof of (2_n). In this way, we see that there exists a unique additive chacteristic class R such that $\mathrm{Err}(\cdot; p_n; R) = 0$ for all $\mathbb{P}^n$-bundles p_n.

B. Closed Immersions of higher codimension

(B.1) In this section, we deal with a regular closed immersion of higher codimension. Similarly, we use the deformation to the normal cone technique so as to deduce a general immersion to the following situation:

(i) a section of a projective bundle;

(ii) codimension-one closed immersions.

(B.2) Let

$$\begin{array}{ccc}
X & \overset{i_n}{\hookrightarrow} & Z \\
 f \searrow & & \swarrow g \\
 & Y &
\end{array}$$

be a closed immersion $i_n : X \hookrightarrow Z$ of codimension n, smooth over Y via f and g. Similarly, as for i_1 before, for any vector bundle E on X with R as in Section A, set

$$\mathrm{Err}(E; i_n; R) := \mathrm{Err}(E; f; R) - \mathrm{Err}((i_n)_* E; g; R).$$

We here want to show that $\mathrm{Err}(\cdot; p_n; R) = 0$ and $\mathrm{Err}(\cdot; i_1; R) = 0$ implies $\mathrm{Err}(\cdot; i_n; R) = 0$.

(B.3) Let us now suppose that we can reduce the problem for a general closed immersion i_n to the case 1(i) and 1(ii) above, then by definition, in case (i),

$$\mathrm{Err}(E; i_n; R) = \mathrm{Err}(E; X \xrightarrow{f} Y; R) - \mathrm{Err}((i_n)_* E; \mathbb{P}^n_X \to X \to Y; R).$$

Note that now i_n is simply a section of $\mathbb{P}^n_X \to X$, hence with the same proof as for Lemma 4.B.7.1, (see e.g. Remark 5.4,) we have $\mathrm{Err}(\cdot; i_n; R) = 0$ for i_n a section of a projective bundle. Note also that by Proposition 5.A.7, Err is zero for any closed immersion of codimension one, so we should find a way to deduce a general closed immersion to a section of a $\mathbb{P}^n$-bundle and codimension one closed immersions.

(B.4) To deduce the case of an arbitrary closed immersion to 1(i) and 1(ii) above, we use deformation to the normal cone theory as usual. For this, recall the following basic fact concerning the deformation to the normal cone.

Let $f : X \to Y$ and $g : Z \to Y$ be two smooth, proper morphisms of compact Kähler manifolds and $i : X \to Z$ be a codimension n closed immersion over Y, i.e., i is a closed immersion of codinemsion n such that $f = g \circ i$. Then we have the following standard construction of the deformation to the normal cone.

Denote by $\pi : W := B_{X \times \{\infty\}} Z \times \mathbb{P}^1 \to Z \times \mathbb{P}^1$, where $B_{X \times \{\infty\}} Z \times \mathbb{P}^1$ denotes the blowing-up of $Z \times \mathbb{P}^1$ along $X \times \{\infty\}$. Denote the exceptional divisor by $\mathbb{P}$. It is well-known that the map $q_W : W \to \mathbb{P}^1$, obtained by composing π with the projection $q : Z \times \mathbb{P}^1 \to \mathbb{P}^1$, is flat, and that for $z \in \mathbb{P}^1$:

$$q^{-1}(z) = \begin{cases} Z, & \text{for } z \neq \infty, \\ \mathbb{P} \cup B_X Z, & \text{for } z = \infty. \end{cases}$$

Here $B_X Z$ denotes the blowing-up of Z along X. By the construction, $\mathbb{P} \cap B_X Z$ is the exceptional diviosr of $B_X Z$. In particular, $\mathbb{P}^n = \mathbb{P}_X(N_{i_n} \oplus \mathcal{O}_X)$ with N_{i_n} the normal bundle corresponding to i_n.

Denote by $I : X \times \mathbb{P}^1 \hookrightarrow W$ the induced codimension n closed embedding. Easily we see that the image of I does not intersect with $B_X Z$, and the image $X \times \{\infty\}$ in W is a section of $\mathbb{P}$.

Lemma. *With the same notation as above, the following two morphisms*

$$X \stackrel{i_\infty}{\hookrightarrow} \mathbb{P}_X(N_{i_n} \oplus \mathcal{O}_X) \stackrel{j_\infty}{\hookrightarrow} W \qquad \text{and} \qquad X \stackrel{i_0}{\hookrightarrow} W_0 \stackrel{j_0}{\hookrightarrow} W$$

induce the same morphism for K-groups.

Proof. Clearly, as W is flat over the rational curve $\mathbb{P}^1$, and K-group is compactible with rational equivalence on cycles by the fact that there is a $\mathbb{Q}$-isomorphism between K-groups and Chow groups, we see that $X \hookrightarrow W_t$ for all $t \in \mathbb{P}^1$ induce the same map on K-groups.

Now let $0 \to E. \to I_*F \to 0$ be any resolution of a vector bundle on X, (viewed as a vector bundle on $X \times \mathbb{P}^1$.) Then by the flatness of $W \to \mathbb{P}^1$, the restrictions of resolutions $0 \to E. \to I_*F \to 0$ to W_t for all $t \in \mathbb{P}^1$ are all exact. On the other hand, by the construction, $X \times \mathbb{P}^1$ is disjoint from the component $B_X Z$ in W_∞, so we see that on $B_X Z$, such a restriction is acyclic. That is to say, it gives zero element in $K(B_X Z)$ and as well as in $K(\mathbb{P} \cap B_X Z)$. Thus by definition, we see that the K-push-forward for $X \to W_\infty \to W$ may be simply calculated by the one induced via $X \to \mathbb{P} \to W$. This completes the proof of the lemma.

In this way, by definition, we see that

$$\mathrm{Err}(E; j_\infty \circ i_\infty; R) = \mathrm{Err}(E; j_0 \circ i_0; R).$$

By definition again, we know that

$$\mathrm{Err}(E; j_\infty \circ i_\infty; R) = \mathrm{Err}(E; i_\infty; R) + \mathrm{Err}(i_{\infty *}E; j_\infty; R)$$

and

$$\mathrm{Err}(E; j_0 \circ i_0; R) = \mathrm{Err}(E; i_0; R) + \mathrm{Err}(i_{0*}E; j_0; R).$$

Thus to complete the proof, it is sufficient to prove that

$$\mathrm{Err}(\cdot; i_\infty; R) \equiv 0,$$

$$\mathrm{Err}(\cdot; j_\infty; R) \equiv 0,$$

and

$$\mathrm{Err}(\cdot; j_0; R) \equiv 0.$$

Note that each of the three closed immersions, i_∞, j_0 and j_∞, is either a codimension-one closed immersion or a section of a $\mathbb{P}^n$-bundle, i.e., they belong exactly to 1(i)

and 1(ii) above. So we see that $\mathrm{Err}(\cdot; i_n; R) \equiv 0$ for any closed immersion i_n of codimension n.

C. The Proof of Uniqueness Theorems

(C.1) Clearly, the weak uniqueness theorem is a direct consequence of the strong uniqueness theorem, so it is sufficient to prove the later one.

(C.2) For doing so, factor $f : X \to Y$ as a regular closed immersion $i_m : X \to Z$ followed by a projection $p_n : \mathbb{P}^n_Y \to Y$. Then by definition, for any vector bundle E on X,

$$\begin{aligned} &\mathrm{Err}(E; f; R) \\ =&\mathrm{Err}(E; i_m; R) + \mathrm{Err}(i_*E; p_n; R) \\ =&0 + \mathrm{Err}(i_*E; p_n; R) \qquad \text{(by (B.4))} \\ =&0 + 0 \qquad \text{(by (A.3) and (A.4))} \\ =&0. \end{aligned}$$

This together with Proposition 5.A.3 implies that

$$\mathrm{Err}(\cdot; f; R) \equiv 0.$$

Thus by definition, we have

$$\mathrm{ch}'_{\mathrm{BC}}(E, \rho; f, \tau_f) = \mathrm{ch}_{\mathrm{BC}}(E, \rho; f, \tau_f) + f_*\Big(\mathrm{ch}(E) \cdot \mathrm{td}(T_f) \cdot R(T_f)\Big)$$

for any smooth proper morphism f and any f-acyclic vector bundle E. This then surely completes the proof of the uniqueness theorems.

7. Existence of Relative Bott-Chern Secondary Characteristic Classes

A) Arithmetic intersection and arithmetic characteristic classes

B) An Effective Construction of Relative Bott-Chern Secondary Characteristic Classes

In this chapter, we prove a weak version of the existence of relative Bott-Chern secondary characteristic classes by effectively constructing some classes of differential forms, which is sufficient and necessary for our application to the arithmetic Grothendieck-Riemann-Roch theorem.

A. Arithmetic intersection and arithmetic characteristic classes

(A.1) In this section, we first recall the theory of arithmetic intersection and arithmetic characteristic classes developed by Arakelov [Ar1,2], Deligne [De2] and Gillet-Soulé [GS1,2]. All results in this section are mainly due to [GS1,2]. The notable difference is that here we only work over $\mathbb{C}$. So some modifications are needed. (Contary to the general principle used in this paper, in this section, we will only tell the reader how an arithmetic intersection and arithmetic characteristic classes are introduced, instead of indicating a more sound why.)

Let X be a complex compact manifold of dimension d. Denote the space of differential forms of degree n on X by $A^n(X) := \oplus_{p+q=n} A^{p,q}(X)$. There are natural boundary morphisms $\partial : A^{p,q}(X) \to A^{p+1,q}(X), \bar{\partial} : A^{p,q}(X) \to A^{p,q+1}(X)$, and the usual differential $d : A^n(X) \to A^{n+1}(X)$. We say that a linear function T on $A^n(X)$ is a *current*, if T is continuous in the sense of Schwartz: for any sequence $\{\omega_r\} \subset A^n(X)$ with the supports contained in certain fixed compact subset K, $T(\omega_r) \to 0$ if all the coefficients of ω_r together with their derivatives tend uniformly to zero when $r \to \infty$. The set of currents forms a topological dual space $A(X)^*$ of $A(X)$. Denote by $D_n(X) := A^n(X)^*$. There is a natural decomposition $D_n(X) = \oplus_{p+q=n} D_{p,q}(X)$, where $D_{p,q}(X)$ is the dual of $A^{p,q}(X)$. It is convenient to set $D^{p,q}(X) := D_{d-p,d-q}(X)$. Then $\partial, \bar{\partial}$, and d induce morphisms $\partial', \bar{\partial}', d'$ from $D^{p,q}$ to $D^{p+1,q}(X)$, $D^{p,q+1}(X)$, and $D^{p+1,q+1}(X)$ respectively, e.g., $(\partial' T)(\alpha) := T(\partial\alpha)$.

Examples. (i) There is a natural inclusion

$$\begin{array}{ccc} A^{p,q}(X) & \hookrightarrow & D^{p,q}(X) \\ \omega & \mapsto & [\omega] \end{array},$$

where $[\omega](\alpha) := \int_X \omega \wedge \alpha$ for any $\alpha \in A^{d-p,d-q}(X)$. We say that a current T is *smooth* if there exists a smooth form ω such that $T = [\omega]$. In particular, if $p+q=n$, it follows by Stokes' theorem that $[d\omega](\alpha) = \int_X d\omega \wedge \alpha = \int_X d(\omega \wedge$

$\alpha) - \int_X (-1)^n \omega \wedge d\alpha = (-1)^{n+1} \int_X \omega \wedge d\,\alpha = (-1)^{n+1}(d'[\omega])(\alpha)$. Therefore, we let $\partial, \bar\partial, d$ on the currents be $(-1)^{n+1}\partial'$, $(-1)^{n+1}\bar\partial'$, $(-1)^{n+1}d'$ respectively, and let $d^c := \frac{1}{4\pi i}(\partial - \bar\partial)$. Then $dd^c = -\frac{1}{2\pi i}\partial\bar\partial$ is a real operator, and we have the following commutative diagram:

$$\begin{array}{ccc} A^{p,q}(X) & \hookrightarrow & D^{p,q}(X) \\ \partial\downarrow & & \downarrow\partial \\ A^{p+1,q}(X) & \hookrightarrow & D^{p+1,q}(X). \end{array}$$

(ii) Let $i : Y \hookrightarrow X$ be an irreducible subvariety of codimension p. We get a current $\delta_Y \in D^{p,p}(X)$ by letting $\delta_Y(\alpha) := \int_{Y^{\rm ns}} i^*\alpha$ for any $\alpha \in A^{d-p,d-p}(X)$. Here $Y^{\rm ns}$ denotes the non-singular locus of Y. We call this current the *Dirac symbol* of Y.

Concerning the relations between smooth differential forms and currents, we have the following well-known facts:

(i) With the boundary morphisms ∂, $\bar\partial$, d, the cohomologg goups of X for differential forms are isomorphic to these for currents.

(ii) Let γ be a current on X such that $dd^c\gamma$ is smooth. Then there exist currents ω, α, β such that $\gamma = \omega + \partial\alpha + \bar\partial\beta$, with ω smooth.

(iii) As a current, if ω is smooth and $\omega = \partial u + \bar\partial v$, then there exist smooth currents α, β such that $\omega = \partial\alpha + \bar\partial\beta$.

(iv) If X is a Kähler manifold, and $\eta \in D^{p,q}(X)$, $p, q \geq 1$, is d-closed and is either d, ∂, $\bar\partial$ exact. Then there exists $\gamma \in D^{p-1,q-1}(X)$ such that $dd^c\gamma = \eta$. In particular, if $\eta = 0$, we may choose $\gamma = \omega + \partial\alpha + \bar\partial\beta$ with ω a harmonic form.

(A.2) Let Y be a codimension p analytic subvariety of X. Following [GS1], we say that a current $g \in D^{p-1,p-1}(X)$ is a *Green's current* of Y if $dd^c g = [\omega] - \delta_Y$ for some $\omega \in A^{p,p}(X)$. It is well-known that if X is a Kähler manifold, then Green's currents for analytic subvarieties of X exist. And if g_1 and g_2 are two Green's currents for Y, then, by A.2(ii), $g_1 - g_2 = [\eta] + \partial S_1 + \bar\partial S_2$, where $\eta \in A^{p-1,p-1}(X)$.

Example. (**The Poincaré-Lelong equation**) Let (L, ρ) be a hermitian line bundle on X and s a non-zero meromorphic section of L. Then $-\log|s|^2_\rho \in L^1(X)$, and hence induces a distribution $[-\log|s|^2_\rho] \in D^{0,0}(X)$. Moreover,

$$dd^c[-\log|s|^2_\rho] = [c_1(L,\rho)] - \delta_{{\rm div}(s)}.$$

So, $[-\log|s|^2_\rho]$ is a Green's current of ${\rm div}(s)$, the divisor of s.

(A.3) From now on, assume that X is a projective complex manifold. For any irreducible subvariety Y, following [GS1], we say a smooth form α on $X - Y$ has

logarithmic growth along Y, if there exists a proper morphism $\pi : \tilde{X} \to X$ such that $E := \pi^{-1}(Y)$ is a divisor with normal crossings, $\pi : \tilde{X} - E \simeq X - Y$ and α is the direct image of a form β on $\tilde{X} - E$ by π with the following property: Near each $x \in \tilde{X}$, let $z_1 \ldots z_k = 0$ be a local defining equation of E. Then, there exists d-closed smooth forms α_i and a smooth form γ such that $\beta = \sum_{i=1}^{k} \alpha_i \log|z_i|^2 + \gamma$.

Obviously, such an α is always locally integrable on X, and hence defines a current $[\alpha]$, which is the direct image by π of the current $[\beta]$.

By [GS1], we know that for every irreducible subvariety $Y \subset X$, there exists a smooth form g_Y on $X - Y$ with logarithm growth along Y such that $[g_Y]$ is a Green's current for Y. (This may be viewed as a generalization of the Poincaré-Lelong equation.) Moreover, if α is a form on $X - Y$ with logathmic growth along Y, then

(i) if $f : X' \to X$ is a morphism of smooth projective varieties such that $f^{-1}(Y)$ does not contain any component of X', then the form $f^*(\alpha)$ is of logarithmic growth along $f^{-1}(Y)$;

(ii) $d[\alpha] = [d\alpha]$.

(A.4) We introduce now the arithmetic Chow groups and their cohomological properties following [GS1].

Let X be a regular projective variety over $\mathbb{C}$. Set

$$\tilde{A}^{p,p}(X) := A^{p,p}(X)/(\mathrm{Im}\partial + \mathrm{Im}\bar{\partial}); \quad \tilde{A}(X) := \oplus_p \tilde{A}^{p,p}(X);$$

$$\tilde{D}^{p,p}(X) := D^{p,p}(X)/(\mathrm{Im}\partial + \mathrm{Im}\bar{\partial}); \quad \tilde{D}(X) := \oplus_p \tilde{D}^{p,p}(X).$$

Denote by $Z^p(X)$ the free group generated by subvarieties of X. We say that an element $(Z, g_Z) \in Z^p(X) \oplus \tilde{D}^{p-1,p-1}(X)$ is an *arithmetic p-cycle* if g_Z is a Green's current of Z, i.e. $dd^c g_z = \omega(Z, g_Z) - \delta_Z$ for some $\omega_Z := \omega(Z, g_Z) \in A^{p,p}(X)$. Denote by $Z^p_{\mathrm{Ar}}(X)$ the abelian group generated by arithmetic p-cycles.

Next, define arithmetic rational equivalence among arithmetic cycles. Let $i : Y \hookrightarrow X$ be a subvariety of codimension $p-1$. Then, there is a resolution of singularities of Y, $\pi : \tilde{Y} \to Y$ with π proper. Thus any rational function $f \in k(Y)^*$ defines a rational function $\tilde{f}$ on $\tilde{Y}$ such that $\log|\tilde{f}|^2$ is L^1 on $\tilde{Y}$. Hence $\tilde{f}$ is contained in $D^{0,0}(\tilde{Y})$. Let $\tilde{i} : \tilde{Y} \to X$ be the natural induced morphism, then $\tilde{i}_*[\log|\tilde{f}|^2] \in D^{p-1,p-1}(X)$, and is independent of the choice of $\tilde{Y}$. Hence we may simply denote

it by $i_*[\log|f|^2]$. Moreover, by the Poincaré-Lelong equation in Example A.2,

$$\mathrm{div}_{\mathrm{Ar}}(f) := (\mathrm{div}(f), -i_*[\log|f|^2]) \in Z^p_{\mathrm{Ar}}(X),$$

and is defined to be *arithmetically rationally equivalent to zero*. Let $R^p_{\mathrm{Ar}}(X)$ be the subgroup of $Z^p_{\mathrm{Ar}}(X)$ generated by $\mathrm{div}_{\mathrm{Ar}}(f)$ for $f \in k(W)^*$, with W any codimension-$(p-1)$ subvariety. Define the *p-th arithmetic Chow group*, denoted by $\mathrm{CH}^p_{\mathrm{Ar}}(X)$, to be the quotient group $Z^p_{\mathrm{Ar}}(X)/R^p_{\mathrm{Ar}}(X)$. Set $\mathrm{CH}_{\mathrm{Ar}}(X) := \oplus_p \mathrm{CH}^p_{\mathrm{Ar}}(X)$.

(A.5) Let X be a smooth projective variety over $\mathbb{C}$, and $Y \subset X$ a closed irreducible subvariety and $f : Z \to X$ a proper morphism of irreducible projective varieties such that $f(Z) \not\subset Y$. Let g_Y be a Green's current of Y with logarithmic growth along Y associated to $(\tilde{X}, E)$ as in A.3. Denote the associated current by $[g_Y]$. Then by resolving the singularities of Z, we may construct a commutative diagram

$$\begin{array}{ccc} \tilde{Z} & \overset{j}{\to} & \tilde{X} \\ p\downarrow & \searrow q & \downarrow \pi \\ Z & \overset{f}{\to} & X, \end{array}$$

such that $D = j^{-1}(E)$ is a divisor with normal crossings, $\tilde{Z}$ is projective and smooth, and p is birational. By A.3(i), q^*g_Y is of logarithmic growth along $q^{-1}(Y)$, so it is integrable and $[g_Y] \wedge \delta_Z := q_*[q^*g_Y]$ defines a current in X. Furthermore, if g_Z is an arbitrary Green's current of Z, we define the *-product of $[g_Y]$ and g_Z by

$$[g_Y] * g_Z := [g_Y] \wedge \delta_Z + [dd^c g_Y + \delta_Y] \wedge g_Z.$$

One checks that the following holds;

(i) If Y and Z intersect properly, i.e., $Y \cap F = \cup_i S_i$ with $\mathrm{codim}_X S_i = \mathrm{codim}_X Y + \mathrm{codim}_X Z$, and as algebraic cycles, $[Y][Z] = \sum_i \mu_i S_i$, then $dd^c([g_Y] * g_Z) = [\omega_Y \wedge \omega_Z] - \sum_k \mu_k \delta_{S_k}$.

(ii) For any two Green's currents g_Y, g'_Y of Y with logarithmic growth, as an element of $\tilde{D}(X)$, $[g_Y] * g_Z = [g'_Y] * g_Z$ for any Green's current g_Z. Hence, we may also define the *-product among general Green's currents. In particular, $g_Y * g_Z = g_Z * g_Y$, and $(g_Y * g_Z) * g_W = g_Y * (g_Z * g_W)$ whenever the products make sense.

(A.6) Easily, we have the following morphisms involving $\mathrm{CH}_{\mathrm{Ar}}(X)$;

(i) $\zeta : \mathrm{CH}^p_{\mathrm{Ar}}(X) \to \mathrm{CH}(X), \quad (Z, g_Z) \mapsto Z.$

(ii) $a : \tilde{A}^{p-1,p-1}(X) \to \mathrm{CH}^p_{\mathrm{Ar}}(X), \quad \alpha \mapsto (0, \alpha).$

(iii) $\omega : \mathrm{CH}^p_{\mathrm{Ar}}(X) \to A^{p,p}(X), \quad (Z, g_Z) \mapsto dd^c g_Z + \delta_Z.$

One checks that we have the following exact exact sequence

$$\tilde{A}^{p-1,p-1}(X) \xrightarrow{a} \mathrm{CH}^p_{\mathrm{Ar}}(X) \xrightarrow{\zeta} \mathrm{CH}^p(X) \to 0.$$

(A.7) Now we are ready following [GS1] to introduce an arithmetic intersection theory for regular projective varieties over $\mathbb{C}$.

Theorem. ([GS1]) *Let X be a regular projective variety over $\mathbb{C}$. Then*

(i) for each pair of natural numbers (p, q), there is a pairing

$$\begin{array}{ccc} CH^p_{\mathrm{Ar}}(X) \otimes CH^q_{\mathrm{Ar}}(X) & \to & CH^{p+q}_{\mathrm{Ar}}(X)_{\mathbb{Q}} \\ \alpha \otimes \beta & \mapsto & \alpha\beta. \end{array}$$

The pairing is uniquely determined by the following property: If Y and Z are subvarieties of X which intersect properly, and g_Y and g_Z are Green's currents for Y and Z, then

$$([Y], g_Y)([Z], g_Z) := ([Y][Z], g_Y * g_Z);$$

(ii) the product above makes $CH_{\mathrm{Ar}}(X)_{\mathbb{Q}} := \oplus_p CH^p_{\mathrm{Ar}}(X)_{\mathbb{Q}}$ a commutative, associative $\mathbb{Q}$-algebra;

(iii) the natural morphism

$$(\zeta, \omega) : \oplus_p CH^p_{\mathrm{Ar}}(X)_{\mathbb{Q}} \to \oplus_p (CH^p(X) \oplus Z^{p,p}(X))_{\mathbb{Q}}$$

is a $\mathbb{Q}$-algebra homomorphism, where $Z^{p,p}(X) :=$ the closed forms in $A^{p,p}(X)$. Moreover, $a(\phi) \cdot (Z, g) = a(\phi \wedge \omega(Z, g))$ for all $\phi \in \tilde{A}(X)$ and $(Z, g) \in CH_{\mathrm{Ar}}(X)$, and in particular, $a(\oplus_p H^{p,p}(X))$ is a square zero ideal in $CH_{\mathrm{Ar}}(X)_{\mathbb{Q}}$.

(A.8) Let $f : X \to Y$ be a smooth morphism of regular projective varieties over $\mathbb{C}$. Then for any subvariety Z in Y, if g_Z is a Green's curent of Z, $f^* g_Z$ is a Green's current for $f^{-1}(Z)$. One checks that the arithmetic rational equivalence is compactible with such a pull-back. Hence we have a well-defined pull-back morphism $f^* : \mathrm{CH}^p_{\mathrm{Ar}}(Y) \to \mathrm{CH}^p_{\mathrm{Ar}}(X)$, which induces a $\mathbb{Q}$-algebra morphism $\mathrm{CH}_{\mathrm{Ar}}(Y)_{\mathbb{Q}} \to \mathrm{CH}_{\mathrm{Ar}}(X)_{\mathbb{Q}}$

We may also construct a puh-forward morphism. This is done as follows. First construct a map from $Z^p_{\mathrm{Ar}}(X)$ to $Z^{p-r}_{\mathrm{Ar}}(Y)$ as follows, where r denotes the relative dimension of f: Let $(Z, g_Z) \in Z^p_{\mathrm{Ar}}(X)$, with Z irreducible, i.e. $Z = \overline{\{z\}}$ with z the generic point of Z. Set

$$f_*(Z) := \begin{cases} [k(z) : k(f(z))]\, \overline{\{f(z)\}}, & \text{if } \dim f(z) = \dim z; \\ 0, & \text{otherwise.} \end{cases}$$

Then, for Green's currents, we know that for any $\eta \in \tilde{A}^{\dim Y-p,\dim Y-p}(X)$,

$$
\begin{aligned}
(f_*\delta_Z)(\eta) =& \delta_Z(f^*\eta) = \int_Z f^*\eta = \int_Z f^*(\eta|_{f(Z)}) \\
=& \begin{cases} \deg(Z/f(Z)) \int_{f(Z)} \eta, & \text{if } Z \to f(Z) \text{ is finite;} \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

Hence $f_*\delta_Z = \delta_{f_*(Z)}$, and

$$dd^c(f_*g_Z) = [f_*\omega_Z] - \delta_{f_*(Z)}.$$

That is, f_*g_Z defines a Green's current of $f_*(Z)$. Therefore we may set

$$f_*(Z, g_Z) := (f_*Z, f_*g_Z) \in Z_{\mathrm{Ar}}^{p-r}(Y).$$

Furthermore, it is not difficult to check that this definition is compatible with the arithmetic rational equivalence, and hence we get a push-out morphism f_* for arithmetic Chow groups $f_* : \mathrm{CH}_{\mathrm{Ar}}^p(X) \to \mathrm{CH}_{\mathrm{Ar}}^{p-r}(Y)$, which induces a $\mathbb{Q}$-algebra morphism $\mathrm{CH}_{\mathrm{Ar}}(X)_{\mathbb{Q}} \to \mathrm{CH}_{\mathrm{Ar}}(Y)_{\mathbb{Q}}$. Moreover, we have the *projection formula*

$$f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta).$$

One may also introduce a pull-back morphism for arithmetic Chow rings with respect to closed immersions $i : X \hookrightarrow Z$ of regular projective varieties over $\mathbb{C}$. In fact if W is a subvariety of Z which properly intersects with X, then it is easily to check that for any Green's current g_W of W, i^*g_Z is a Green's current of $W \cap Z$. Hence, $i^*(W, g_W) \in \mathrm{CH}_{\mathrm{Ar}}(Z)$. For general arithmetic cycles, one may use the arithmetic intersection to define their pull-back onto Z. So a suitable Chow type moving lemma is needed. All in all, as what does in [GS1], for i, we finally can define a $\mathbb{Q}$-algebra morphism $i^* : \mathrm{CH}_{\mathrm{Ar}}(Z)_{\mathbb{Q}} \to \mathrm{CH}_{\mathrm{Ar}}(X)_{\mathbb{Q}}$ which coincides i^* defined above for arithmetic cycles whose corresponding algebraic cycles are properly intersect with Z.

Thus, in particular, if $f : X \to Y$ is a morphism between regular projective varieties over $\mathbb{C}$, then write f as a composition of a closed immersions i and a smooth moprphism p. Set $f^* := p^* \circ i^* : \mathrm{CH}_{\mathrm{Ar}}(Y) \to \mathrm{CH}_{\mathrm{Ar}}(X)$. One checks that such an f^* does not depend on the decomposition $f = p \circ i$.

Moreover, if $f : X \to Y$ is smooth and $g : Y' \to Y$ is proper morphism among regular projective varieties over $\mathbb{C}$. Denote by $g_f : X \times_Y Y' \to X$ and $f_g : X \times_Y Y' \to Y'$ the projections as before. Then for any element $(Z, g) \in \mathrm{CH}_{\mathrm{Ar}}(X)$, we have $g^*f_*(Z, g) = (f_g)_*(g_f)^*(Z, g) \in \mathrm{CH}_{\mathrm{Ar}}(Y')$.

(A.9) Let X be regular projective variety over $\mathbb{C}$. Then, following [GS2], we define the *arithmetic K-group* $K_{\mathrm{Ar}}(X)$ as the abelian group generated by hermitian vector bundles (E,ρ) on X and $\eta \in \tilde{A}(X)$ modulo the subgroup generated by the following relations: For any short exact sequence of vector bundles on X,

$$E.: \qquad 0 \to E_1 \to E_2 \to E_3 \to 0,$$

let ρ_i be hermitian metrics on E_i, then

$$((E_1,\rho_1);\eta_1) + ((E_3,\rho_3);\eta_3) = ((E_2,\rho_2); -\mathrm{ch}_{\mathrm{BC}}(E.,\rho_1,\rho_2,\rho_3) + \eta_1 + \eta_3).$$

Here $\mathrm{ch}_{\mathrm{BC}}(E.;\rho.)$ denotes the classical Bott-Chern secondary characteristic classes associated with $(E.,\rho.)$ on X with respect to ch. (See e.g., Remark 1.1.1.)

To motivate the definition of arithmetic characteristic classes, we start with an example. Let (L,ρ) be a hermitian line bundle on X. Then for any non zero section s of L, by the Poincaré-Lelong equation in Example A.2, $(\mathrm{div}(s), -[\log |s|_\rho^2])$ is an element of $\mathrm{CH}^1_{\mathrm{Ar}}(X)$. Define $c_{\mathrm{Ar},1}(L,\rho)$ as the class of this element in the arithmetic Chow ring. From here, as one may imagine, in general, we may use the splitting principle together with the classical Bott-Chern secondary characteristic classes to construct arithmetic characteristic classes for hermitian vector bundles.

(A.10) Let B be a subring of real number field $\mathbb{R}$, and let $\phi \in B[[T_1,\ldots,T_n]]$ be a symmetric power series. Then we have the following;

Theorem. ([GS2]) *Associated to every hermitian vector bundle (E,ρ) of rank n on X is an arithmetic characteristic class $\phi_{\mathrm{Ar}}(E,\rho) \in CH_{\mathrm{Ar}}(X)_{\mathbb{Q}}$ which satisfies the following properties:*

(i) If $f: Y \to X$ is a morphism, $f^*(\phi_{\mathrm{Ar}}(E,\rho)) = \phi_{\mathrm{Ar}}(f^*E, f^*\rho)$.

(ii) If $(E,\rho) = (L_1,\rho_1) \oplus \cdots \oplus (L_n,\rho_n)$ is an orthogonal direct sum of hermitian line bundles, $\phi_{\mathrm{Ar}}(E,\rho) = \phi(c_{\mathrm{Ar},1}(L_1,\rho_1),\ldots,c_{\mathrm{Ar},1}(L_n,\rho_n))$.

(iii) In $\tilde{A}(X)$, $\omega(\phi_{\mathrm{Ar}}(E,\rho)) = \phi(E,\rho)$, and in $CH(X)_{\mathbb{Q}}$, $\zeta(\phi_{\mathrm{Ar}}(E,\rho)) = \phi(E)$.

(iv) Let $\quad E.: \quad 0 \to E_1 \to E_2 \to E_3 \to 0 \quad$ be an exact sequence of vector bundles on X together with hermitian metrics ρ_i on E_i for $i = 1,2,3$. Then

$$\phi_{\mathrm{Ar}}(E_2,\rho_2) = \phi_{\mathrm{Ar}}(E_1 \oplus E_3, \rho_1 \oplus \rho_3) + a(\phi_{\mathrm{BC}}(E.,\rho_1,\rho_2,\rho_3)).$$

In particular, we have a well-defined arithmetic Chern characteristic class

$$\mathrm{ch}_{\mathrm{Ar}} : K_{\mathrm{Ar}}(X)_{\mathbb{Q}} \to \mathrm{CH}_{\mathrm{Ar}}(X)_{\mathbb{Q}}.$$

It is a result of Gillet and Soulé that $K_{\mathrm{Ar}}(X)_{\mathbb{Q}}$ admits a so-called λ-ring structure, and $\mathrm{ch}_{\mathrm{Ar}}$ indeed gives a ring isomorphim. We will not recall the details, instead, we state the following equivalent:

Theorem$'$. ([GS2]) *Let X be a regular projective variety over $\mathbb{C}$. Then*

$$ch_{\mathrm{Ar}} : K_{\mathrm{Ar}}(X)_{\mathbb{Q}} \to CH_{\mathrm{Ar}}(X)_{\mathbb{Q}}$$

is a group isomorphism. Moreover,

$$ch_{\mathrm{Ar}}\Big((E,\rho)\otimes(E',\rho')\Big) = \big(ch_{\mathrm{Ar}}(E,\rho)\big)\cdot\big(ch_{\mathrm{Ar}}(E',\rho')\big).$$

(A.11) We end this section by recalling a result due to Faltings.

Let $f : X \to Y$ and $g : Z \to Y$ be two smooth, proper morphisms of compact Kähler manifolds and $i : X \to Z$ be a codimension one closed immersion over Y, i.e., i is a closed immersion of codinemsion one such that $f = g \circ i$. Denote by

$$\pi : W := B_{X\times\{\infty\}}Z \times \mathbb{P}^1 \to Z \times \mathbb{P}^1,$$

the natural projection, where $B_{X\times\{\infty\}}Z \times \mathbb{P}^1$ denotes the blowing-up of $Z \times \mathbb{P}^1$ along $X \times \{\infty\}$. Denote the exceptional divisor by $\mathbb{P}$ of π. We know that the map $q_W : W \to \mathbb{P}^1$, obtained by composing π with the projection $q : Z \times \mathbb{P}^1 \to \mathbb{P}^1$, is flat, and that for $t \in \mathbb{P}^1$:

$$q_W^{-1}(t) = \begin{cases} Z \times \{t\}, & \text{for } t \neq \infty, \\ \mathbb{P} \cup B_X Z, & \text{for } t = \infty. \end{cases}$$

Here $B_X Z$ denotes the blowing-up of Z along X. Moreover, by the construction, $\mathbb{P}$ and $B_X Z$ intersect transversally, and $\mathbb{P} \cap B_X Z$ is the exceptional divisor X on $B_X Z$.

Denote by $I : X \times \mathbb{P}^1 \hookrightarrow W$ the induced codimension one closed embedding. Then the image of I does not intersect with $B_X Z$, and the image $X \times \{\infty\}$ in W is a section of $\mathbb{P}$. Denote the induced fibration $Z \times \{t\} \to Y \times \{t\}$ by g_t for $t \neq \infty$ and set g_∞ to be the composition of the projection of $\mathbb{P}$ on X with $(X =)X \times \{\infty\} \to Y \times \{\infty\}(= Y)$. Denote by $f_t : X \times \{t\} \to Y \times \{t\}$ the smooth morphisms induced from f for all $t \in \mathbb{P}^1$.

Let E be a vector bundles on Z. Then we have the following exact sequence of coherent sequences on W;

$$0 \to (\pi\circ p_Z)^* E(B_X Z - X\times\mathbb{P}^1) \to (\pi\circ p_Z)^* E(B_X Z) \to I_* I^*((\pi\circ p_Z)^* E(B_X Z)) \to 0. \tag{7.1}$$

Thus, by the flatness of $q_W : W \to \mathbb{P}^1$, we know that the restriction of (7.1) to the fibers W_t of q_W for all $t\in\mathbb{P}^1$ are exact. Thus, in particular, for each $t\neq\infty$ in $\mathbb{P}^1$, from (7.1), we have the induced exact sequences

$$0 \to E_t(-X) \to E_t \to (i_t)_* i_t^* E_t \to 0 \qquad \text{over} \quad Z\times\{t\} \tag{7.2}$$

with E_t the pull-back of E under the canonical identity $X\times\{t\}\simeq X$. Similarly, for the fiber at ∞, if we set $E(B_X Z)\big|_{\mathbb{P}} := E_\infty$, $E(B_X Z)\big|_{B_X Z} =: E'_\infty$ and $E(B_X Z - X\times\mathbb{P}^1)\big|_{B_X Z} =: E''_\infty$. Then $E(B_X Z - X\times\mathbb{P}^1)\big|_{\mathbb{P}} = E_\infty(-X)$, and (7.1) splits into two exact sequences

$$0 \to E_\infty(-X) \to E_\infty \to (i_\infty)_* i_\infty^* E_\infty \to 0 \qquad \text{over} \quad \mathbb{P} \tag{7.3}$$

and

$$0 \to E'_\infty \to E''_\infty \to 0 \to 0 \qquad \text{over} \quad B_X Z. \tag{7.4}$$

In particular, we see that on $B_X Z$, $E'_\infty = E''_\infty$.

Choose a hermitian metric τ_W on T_W, the tangent bundle of W. Then τ_W naturally induces hermitian metrics τ_t on T_{g_t} for all $t\in\mathbb{P}^1$. Similarly, τ_W induces a hermitian metric τ_G on $T_G(-\log\infty)$, the logarithmic relative tangent bundle associated to the morphism $G : W \to Y\times\mathbb{P}^1$, which may be naturally embedded in T_W (see e.g. [De1]).

Fix hermitian metrics ρ and ρ' on E and on $E(-X)$ respectively. Use the same notation to denote the pull-back of (E,ρ) onto W. Choose the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}^1}(\infty)$ and a metric on $\mathcal{O}_W(-X\times\mathbb{P}^1)$ such that in a neighborhood U of $B_X Z$, which is away from $X\times\mathbb{P}^1$, the natural isomorphism $\mathcal{O}_W(-X\times\mathbb{P}^1)\simeq\mathcal{O}_W$ induces an isometry, once we put the trivial metric on $\mathcal{O}_W$. Denote these final induced metrics on $E(B_X Z)$ and $E(B_X Z - X\times\mathbb{P}^1)$ by $D\rho$ and $D\rho'$ respectively.

Denote the induced metrics via restriction to E_t and $E_t(-X)$ (resp. to E'_∞ and E''_∞ on $B_X E$) by ρ_t and ρ'_t respectively for all $t\in\mathbb{P}^1$, (resp. ρ''_∞ and ρ'''_∞). Easily, we see that $\rho_0=\rho$ and $\rho'_0=\rho'$, and $(E'_\infty,\rho''_\infty)$ is isomorphic to $(E''_\infty,\rho'''_\infty)$ by the construction. In this way,

$$(E(-X),\rho') \hookrightarrow (E,\rho) \qquad \text{on} \quad Z$$

is deformed to

$$(E_\infty(-X),\rho'_\infty)\hookrightarrow(E_\infty,\rho_\infty)\quad\text{on}\quad\mathbb{P}$$

(and

$$(E'_\infty,\rho''_\infty)\simeq(E''_\infty,\rho'''_\infty)\qquad\text{on}\quad B_XZ).$$

Proposition. ([F]) *With the same notation as above, for all $t\in\mathbb{P}^1$,*

$$\begin{aligned}&\Big(ch_{\mathrm{Ar}}(E_t,\rho_t)-ch_{\mathrm{Ar}}(E_t(-X),\rho'_t)\Big)\cdot td_{\mathrm{Ar}}(T_t,\tau_{g_t})\\=&i_t^*\Big(\Big(ch_{\mathrm{Ar}}(E,D\rho)-ch_{\mathrm{Ar}}(E,D\rho)\Big)\cdot td_{\mathrm{Ar}}(T_G(-\log\infty),\tau_G)\Big).\end{aligned}$$

Proof. Surely, by (A.8), i_t^* is well-defined. So at least all terms make perfect sense as they stand.

The proof is rather formal but standard. The key points are that $E-E(-X\times\mathbb{P}^1)$ is supported only on $X\times\mathbb{P}^1$; that $X\times\mathbb{P}^1$ does not intersect B_XZ where the hermitian exact sequence splits. As a direct consequence, in the calculation, we may only pay our attention on $X\times\mathbb{P}^1$, while pay no attention on B_XZ. With this in mind, certainly, we may also simply view (T_0,τ_{g_0}) (resp. $(T_\infty,\tau_{g_\infty})$) as the restriction of the $(T_G(-\log\infty),\tau_G)$. (Recall that the induced metric on T_∞ from τ_G is singular, but the singularity is concentrated on $\mathbb{P}\cap B_XZ$, which is away from $X\times\mathbb{P}^1$ by the construction.) To be more precise, in practice, we need the following preparation.

First, following [F], we give a generalization of arithmetic intersection introduced in A.7. Instead of working over a single regular projective variety over $\mathbb{C}$, we are now working over a triple $(A;B;C)$. Here A and C are regular projective varieties over $\mathbb{C}$, $A\subset B\subset C$ with B an open subset of C, i.e., B an open neighborhood of A in C. Define a relative arithmetic Chow group $\mathrm{CH}^{A,B}_{\mathrm{Ar}}(C)$ by setting it to be the quotient of the group generated by arithmetic cycles (S,g_S) of C with $S\subset A$, $\mathrm{Supp}(g_S)\subset B$, modulo the subgroup generated by arithmetic cycles defined by rational functions on cycles in A, together with the forms $\partial\alpha+\bar\partial\beta$, where α and β are currents with support in A. We point out that even by tensoring with $\mathbb{Q}$, the resulting space $\mathrm{CH}^{A,B}_{\mathrm{Ar}}(C)_{\mathbb{Q}}$ only admits a group structure.

Nevertheless, we may introduce a natural $\mathrm{CH}_{\mathrm{Ar}}(C)_{\mathbb{Q}}$-module structure on $\mathrm{CH}^{A,B}_{\mathrm{Ar}}(C)_{\mathbb{Q}}$. Indeed, for an element $\omega\in\tilde A(C)$, it is clear that for any element $(S,g_S)\in\mathrm{CH}^{A,B}_{\mathrm{Ar}}(C)_{\mathbb{Q}}$, the standard arithmetic intersection on C gives an arthmetic cycle $(0,\omega)\cdot(S,g_S)$ on C which then turns out to be an element in $\mathrm{CH}^{A,B}_{\mathrm{Ar}}(C)_{\mathbb{Q}}$ as well. Similarly, for any hermitian line bundle (L,ρ) on C, Theorem A.7, in which the standard arithmetic intersection is introduced, shows that $c_{1,\mathrm{Ar}}(L,\rho)\cdot(S,g_S):=$

$(\mathrm{div}(s), -\log\|s\|_\rho^2)\cdot(S,g_S)$ is again an element in $\mathrm{CH}_{\mathrm{Ar}}^{A,B}(C)_{\mathbb{Q}}$. Here s is a non-zero section of L. Easily, one checks that all the actions satisfies the module axioms. Thus in particular, by Theorem A.10$'$ and splitting principle, we have provd the first part of the following

Lemma. ([F]) *With the same notation as above,*

(i) $CH_{\mathrm{Ar}}^{A,B}(C)_{\mathbb{Q}}$ admits a natural $CH_{\mathrm{Ar}}(C)_{\mathbb{Q}}$-module structure;

(ii) If (E_1,ρ_1) and (E_2,ρ_2) are two hermitian vector bundles on C such that over B, there is an isometry $(E_1,\rho_1)\Big|_B \simeq (E_2,\rho_2)\Big|_B$, then for any arithmetic characteristic class ϕ_{Ar}, and any element $(S,g_S)\in CH_{\mathrm{Ar}}^{A,B}(C)$, we have in $CH_{\mathrm{Ar}}^{A,B}(C)_{\mathbb{Q}}$ and hence in $CH_{\mathrm{Ar}}(A)$,

$$\phi_{\mathrm{Ar}}(E_1,\rho_1)\cdot(S,g_S)=\phi_{\mathrm{Ar}}(E_2,\rho_2)\cdot(S,g_S).$$

Proof. Let us start with the case for hermitian line bundles. In this case, we have the following

Lemma.$'$ *Let $(A_1;B_1;C_1)$ and $(A_2;B_2;C_2)$ be two triples as above such that there is an isomorphism $a: B_1\simeq B_2$ which induces an isomorphism $A_1\simeq A_2$. Then*

(i) there is a natural induced isomorphism

$$CH_{\mathrm{Ar}}^{A_1,B_1}(C_1)\simeq CH_{\mathrm{Ar}}^{A_2,B_2}(C_2);$$

(ii) If there are hermitian line bundles (L_1,ρ_1) and (L_2,ρ_2) on Z_1 and Z_2 respectively such that there is an isometry $(L_1,\rho_1)\Big|_{B_1}\simeq(L_2,\rho_2)\Big|_{B_2}$, then we have the following commutative diagram;

$$\begin{array}{ccc} CH_{\mathrm{Ar}}^{A_1,B_1}(C_1) & \simeq & CH_{\mathrm{Ar}}^{A_2,B_2}(C_2)\\ {\scriptstyle c_{1,\mathrm{Ar}}(L_1,\rho_1)\cdot(\cdot)}\downarrow & & \downarrow{\scriptstyle (\cdot)\cdot c_{1,\mathrm{Ar}}(L_2,\rho_2)}\\ CH_{\mathrm{Ar}}^{A_1,B_1}(C_1) & \simeq & CH_{\mathrm{Ar}}^{A_2,B_2}(C_2). \end{array}$$

Proof. This comes directly from the definition.

Let us now go back to the proof of the lemma. For doing so, we use the flag varieties of E_1 and E_2. Denote by $\mathrm{Flag}_C E_1$ (resp. $\mathrm{Flag}_C E_2$) the complete flag variety of E_1 (resp. E_2), by $\pi_i:\mathrm{Flag}_C E_i\to Z$ the natural projections, and $B_i:=\pi_i^{-1}(B)$, and $A_i:=\pi_i^{-1}(A)$. By our assumption on (E_i,ρ_i) and Lemma$'$(i), we have an isomorphism

$$\mathrm{CH}_{\mathrm{Ar}}^{A_1,B_1}(\mathrm{Flag}_C E_1)\simeq\mathrm{CH}_{\mathrm{Ar}}^{A_2,B_2}(\mathrm{Flag}_C E_2)$$

with which the pull-back of arithmetic cycles $\pi_i^*(S,g_S)$'s are compactible.

But for $i=1,2$, on $\mathrm{Flag}_C E_i$, $\pi_i^*(E_i)$ has a complete filtration by line bundles $L_{i,\alpha}$. Thus we can let any polynomial in the $c_{1,\mathrm{Ar}}(L_{i,\alpha},\rho_{i,\alpha})$ act on the arithmetic cycles in $\mathrm{CH}_{\mathrm{Ar}}^{A_i,B_i}(\mathrm{Flag}_C E_i)$. By Lemma$'$ (ii), at this level, the actions for $i=1$ and for $i=2$ correspond to each other in a unique way. Easily, one sees that so do the actions for elements in $\tilde{A}$, which in particular includes all classical Bott-Chern secondary characteristic classes $\mathrm{ch}_{\mathrm{BC}}$ measuring the change of arithmetic characteristic classes for hermitian vector bundles with respect to the extension of vector bundles and the change of their metrics. Thus, by Theorem A.10, for $\phi_{\mathrm{Ar}}(E_i,\rho_i)$, there exists an operator $Q(c_{1,\mathrm{Ar}}(L_{i,\alpha},\rho_{i,\alpha}))+\Omega_{\mathrm{BC}}$ on $\mathrm{CH}_{\mathrm{Ar}}(\mathrm{Flag}_C E_i)_{\mathbb{Q}}$, where Q is a power series in $c_{1,\mathrm{Ar}}(L_{i,\alpha},\rho_{i,\alpha})$ and Ω_{BC} the multiplication by a classical Bott-Chern secondary characteristic class, such that for all $(S,g_S)\in \mathrm{CH}_{\mathrm{Ar}}^{A,B}(C)$,

$$\phi_{\mathrm{Ar}}(E_i,\rho_i)\,(S,g_S)=\pi_{i*}\Big(\Big((Q(c_{1,\mathrm{Ar}}(L_{i,\alpha},\rho_{i,\alpha})))+\Omega_{\mathrm{BC}}\Big)\,\pi_i^*(S,g_S)\Big).$$

But, by Lemma$'$(ii), we know that

$$\begin{aligned}&\pi_{1*}\Big(\Big((Q(c_{1,\mathrm{Ar}}(L_{1,\alpha},\rho_{1,\alpha})))+\Omega_{\mathrm{BC}}\Big)\,\pi_1^*(S,g_S)\Big)\\ =&\pi_{2*}\Big(\Big((Q(c_{1,\mathrm{Ar}}(L_{2,\alpha},\rho_{2,\alpha})))+\Omega_{\mathrm{BC}}\Big)\,\pi_2^*(S,g_S)\Big),\end{aligned}$$

via the isomorphism

$$\mathrm{CH}_{\mathrm{Ar}}^{A_1,B_1}(\mathrm{Flag}_C E_1)\simeq \mathrm{CH}_{\mathrm{Ar}}^{A_2,B_2}(\mathrm{Flag}_C E_2).$$

Thus in $\mathrm{CH}_{\mathrm{Ar}}^{A,B}(C)_{\mathbb{Q}}$, and hence in $\mathrm{CH}_{\mathrm{Ar}}(C)_{\mathbb{Q}}$, we have

$$\phi_{\mathrm{Ar}}(E_1,\rho_1)\,(S,g_S)=\phi_{\mathrm{Ar}}(E_2,\rho_2)\,(S,g_S)$$

for all elements $(S,g_S)\in \mathrm{CH}_{\mathrm{Ar}}^{A,B}(C)$. This then completes the proof of the lemma.

With this lemma, now the proof of the proposition can be easily deduced. Indeed, first, since $E-E(-X\times\mathbb{P}^1)$ is supported only on $X\times\mathbb{P}^1$ and $X\times\mathbb{P}^1$ does not intersect $B_X Z$ where the hermitian exact sequence splits, $\mathrm{ch}_{\mathrm{Ar}}(E-E(-X\times\mathbb{P}^1),D\rho-D\rho')\in \mathrm{CH}_{\mathrm{Ar}}^{X\times\mathbb{P}^1;W\backslash\overline{U}}(W)_{\mathbb{Q}}$. Secondly, we easily have the functorial properties for the relative arithmetic interesction defined above. Thus, we may apply the lemma as follows: At the fiber over 0, we choose $(A;B;C)$ to be $(X;W\backslash\overline{U}\cap Z;Z)$, (S,g_S) to be $\mathrm{ch}_{\mathrm{Ar}}(E-E(-X\times\mathbb{P}^1),D\rho-D\rho')\Big|_Z$, ϕ_{Ar} to be $\mathrm{td}_{\mathrm{Ar}}$, and (E_1,ρ_1) to be (T_g,τ_g), (E_2,ρ_2) to be $(T_G(-\log\infty),\tau_G)\Big|_Z$; while at the fiber over ∞ or better over $\mathbb{P}$, we choose $(A;B;C)$ to be $(X;W\backslash\overline{U}\cap\mathbb{P};\mathbb{P})$, (S,g_S) to be $(E-E(-X\times\mathbb{P}^1),D\rho-D\rho')\Big|_{\mathbb{P}}$,

ϕ_{Ar} to be $\mathrm{td}_{\mathrm{Ar}}$, and (E_1,ρ_1) to be $(T_\infty,\tau_{g_\infty})$, (E_2,ρ_2) to be $(T_G(-\log\infty),\tau_G)\Big|_{\mathbb{P}}$. All this then completes the proof of the proposition.

B. An Effective Construction of Relative Bott-Chern Secondary Characteristic Classes

(B.1) In (A.6), we see that there is an exact sequence

$$\tilde{A}^{p-1,p-1}(X) \xrightarrow{a} \mathrm{CH}^p_{\mathrm{Ar}}(X) \xrightarrow{\zeta} \mathrm{CH}^p(X) \to 0. \tag{7.5}$$

Therefore the a-image of $\tilde{A}^{p-1,p-1}(X)$ in $\mathrm{CH}^p_{\mathrm{Ar}}(X)$ is a well-defined space. Due to the fact that if $f: Z \to X$ is a morphism of regular projective varieties over $\mathbb{C}$, then the maps appeared in the above exact sequence are compactible with all the pull-back f^* for the corresponding spaces, and hence $a(\tilde{A}^{p-1,p-1}(X))$, a quotient space of $A^{p-1,p-1}(X)$, is as canonical as $A^{p-1,p-1}(X)$ itself. In particular, we may equally use this space, denoted by $\overline{A}^{p-1,p-1}(X)$, to develop the theory of relative Bott-Chern secondary characteristic classes. That is to say, in the axioms stated for relative Bott-Chernsecondary characteristic classes, we may use $\overline{A}$, instead of the original $\tilde{A}$, e.g., we consider the classical and relative Bott-Chern secondary characteristic classes as elements in $\overline{A}$ rather than as in $\tilde{A}$. One checks easily that the previous uniqueness theorems and their proofs work exactly the same way as before.

(B.2) We now give an effective construction for relative Bott-Chern secondary characteristic classes $\mathrm{ch}_{\mathrm{BC}}$ at the level of $\overline{A} := \oplus_p \overline{A}^{p,p}$. That amounts to saying that for all smooth metrized morphisms and all relative acyclic hermitian vector bundles, we should effectively construct some classes in $\overline{A}$, which satisfy the (corresponding modified) six axioms for relative Bott-Chern secondary characteristic classes.

Let $(f: X \to Y; E,\rho; T_f,\tau_f)$ be a properly metrized datum. Moreover, from now on, we always assume that all manifolds are regular projective over $\mathbb{C}$. Then on the direct image, we have a well-defined L^2-metric $L^2(\rho,\tau)$. (Please note that the Kähler conditions in the definition of a properly metrized datum for τ_f guarantee that the L^2-metric is well-defined via the Hodge decomposition.)

Lemma. *With the same notation as above,*

$$f_*\Big(ch_{\mathrm{Ar}}(E,\rho)\cdot td_{\mathrm{Ar}}(T_f,\tau_f)\Big) - ch_{\mathrm{Ar}}\Big(f_*E, L^2(\rho,\tau_f)\Big) \in \overline{A}(Y).$$

Proof. In fact,

$$
\begin{aligned}
&\zeta\Big(f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(E,\rho)\cdot \mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\Big)-\mathrm{ch}_{\mathrm{Ar}}\Big(f_*E,L^2(\rho,\tau_f)\Big)\Big)\\
=&f_*\Big(\zeta\Big(\mathrm{ch}_{\mathrm{Ar}}(E,\rho)\cdot \mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\Big)\Big)-\zeta\Big(\mathrm{ch}_{\mathrm{Ar}}\Big(f_*E,L^2(\rho,\tau_f)\Big)\Big)\\
=&f_*\Big(\zeta\Big(\mathrm{ch}_{\mathrm{Ar}}(E,\rho)\Big)\cdot\zeta\Big(\mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\Big)-\zeta\Big(\mathrm{ch}_{\mathrm{Ar}}\Big(f_*E,L^2(\rho,\tau_f)\Big)\Big)\\
&\qquad\text{(by Theorem A.7(iii))}\\
=&f_*\Big(\mathrm{ch}(E)\cdot\mathrm{td}(T_f)\Big)-\mathrm{ch}\Big(f_*E\Big)\qquad\text{(By Theorem A.10(iii))}\\
=&0\qquad\text{(by the Grothendieck-Riemann-Roch Theorem in Algebraic Geomretry).}
\end{aligned}
$$

Hence, by the exact sequence (7.5), we complete the proof of the lemma.

Now set

$$
\begin{aligned}
&\mathrm{ch}'_{\mathrm{BC}}(E,\rho;f,\tau_f)\\
:=&f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(E,\rho)\cdot \mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\Big)-\mathrm{ch}_{\mathrm{Ar}}\Big(f_*E,L^2(\rho,\tau_f)\Big)\in\overline{A}(Y).
\end{aligned}
$$

We claim that $\mathrm{ch}'_{\mathrm{BC}}(E,\rho;f,\tau_f)$ gives an effective construction of the relative Bott-Chern secondary characteristic classes. So it remains to check that $\mathrm{ch}'_{\mathrm{BC}}(E,\rho;f,\tau_f)$ satisfies six axioms in Chapters 2 and 3.

(B.3) First, for Axiom 1, the downstairs rule, we have the following;

Lemma. *With the same notation as above,*

$$
dd^c\Big(ch'_{\mathrm{BC}}(E,\rho;f,\tau_f)\Big)=f_*\Big(ch(E,\rho)\cdot td(T_f,\tau_f)\Big)-ch\Big(f_*E,L^2(\rho,\tau_f)\Big).
$$

Proof. This may be obtained by the following calculation. Indeed,

$$
\begin{aligned}
&dd^c\Big(\mathrm{ch}'_{\mathrm{BC}}(E,\rho;f,\tau_f)\Big)\\
=&\omega\Big(\mathrm{ch}'_{\mathrm{BC}}(E,\rho;f,\tau_f)\Big)\quad\text{(as }\mathrm{ch}_{\mathrm{BC}}(E,\rho;f,\tau_f)\in\overline{A}(Y))\\
=&\omega\Big(f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(E,\rho)\cdot \mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\Big)-\mathrm{ch}_{\mathrm{Ar}}\Big(f_*E,L^2(\rho,\tau_f)\Big)\Big)\\
&\qquad\text{(by definition)}\\
=&f_*\Big(\omega\Big(\mathrm{ch}_{\mathrm{Ar}}(E,\rho)\cdot \mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\Big)\Big)-\omega\Big(\mathrm{ch}_{\mathrm{Ar}}\Big(f_*E,L^2(\rho,\tau_f)\Big)\Big)\\
=&f_*\Big(\omega\Big(\mathrm{ch}_{\mathrm{Ar}}(E,\rho)\Big)\cdot\omega\Big(\mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\Big)\Big)-\omega\Big(\mathrm{ch}_{\mathrm{Ar}}\Big(f_*E,L^2(\rho,\tau_f)\Big)\Big)\\
&\qquad\text{(by Theorem A.7(iii))}\\
=&f_*\Big(\mathrm{ch}(E,\rho)\cdot\mathrm{td}(T_f,\tau_f)\Big)-\mathrm{ch}\Big(f_*E,L^2(\rho,\tau_f)\Big)\qquad\text{(by Theorem A.10(iii)).}
\end{aligned}
$$

This completes the proof.

(B.4) For Axiom 2, we should prove the following

Lemma. *With the same notation as above, let (F,ρ') be a hermitian vector bundle on Y, then*

$$ch'_{\mathrm{BC}}(E\otimes f^*F,\rho\otimes f^*\rho';f,\tau_f)=ch'_{\mathrm{BC}}(E,\rho;f,\tau_f)\wedge ch(F,\rho').$$

Proof. In fact,

$$\begin{aligned}
&\mathrm{ch}'_{\mathrm{BC}}(E\otimes f^*F,\rho\otimes f^*\rho';f,\tau_f)\\
=&f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(E\otimes f^*F,\rho\otimes f^*\rho')\cdot \mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\Big)-\mathrm{ch}_{\mathrm{Ar}}\Big(f_*(E\otimes f^*F),L^2(\rho\otimes f^*\rho',\tau_f)\Big)\Big)\\
&\qquad\text{(by definition)}\\
=&f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(E,\rho)\cdot f^*(\mathrm{ch}_{\mathrm{Ar}}(F,\rho'))\cdot \mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\Big)-\mathrm{ch}_{\mathrm{Ar}}\Big((f_*(E),L^2(\rho,\tau_f))\otimes(F,\rho')\Big)\\
&\qquad\text{(by Theorem A.10}'\text{ and Theorem A.10(i))}\\
=&f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(E,\rho)\cdot \mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\Big)\cdot\mathrm{ch}_{\mathrm{Ar}}(F,\rho')-\mathrm{ch}_{\mathrm{Ar}}\Big(f_*(E,L^2(\rho,\tau_f)\Big)\cdot\mathrm{ch}_{\mathrm{Ar}}(F,\rho')\\
&\qquad\text{(by projection formula and Theorem A.10}'\text{)}\\
=&\Big(f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(E,\rho)\cdot \mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\Big)-\mathrm{ch}_{\mathrm{Ar}}\Big(f_*(E,L^2(\rho,\tau_f))\Big)\Big)\cdot\mathrm{ch}_{\mathrm{Ar}}(F,\rho')\\
=&\mathrm{ch}'_{\mathrm{BC}}(E,\rho;f,\tau_f)\wedge\omega\Big(\mathrm{ch}_{\mathrm{Ar}}(F,\rho')\Big)\qquad\text{(by Theorem A.7(iii))}\\
=&\mathrm{ch}'_{\mathrm{BC}}(E,\rho;f,\tau_f)\wedge\mathrm{ch}(F,\rho')\qquad\text{(by Theorem A.10(iii)).}
\end{aligned}$$

This completes the proof.

(B.5) For Axiom 3, we should prove the following

Lemma. *With the same notation as above, if $g:Y'\to Y$ is a morphism between regular projective varieties over $\mathbb{C}$, then*

$$g_f^*\Big(ch'_{\mathrm{BC}}(E,\rho;f,\tau_f)\Big)=ch'_{\mathrm{BC}}\Big(g_f^*(E,\rho;f,\tau_f)\Big).$$

Here as in 2.C.2, we have the base change diagram

$$\begin{array}{ccc}
X\times_Y Y' & \overset{g_f}{\to} & X\\
f_g\downarrow & & \downarrow f\\
Y' & \overset{g}{\to} & Y,
\end{array}$$

and define the pull-back of $(E,\rho;f,\tau_f)$ via g_f by

$$g_f^*(E,\rho;f,\tau_f)=(g_f^*(E,\rho);f_g,g_f^*\tau_f).$$

Proof. In fact, By Theorem A.10(i), the functorial property for arithmetic characteristic classes and A.8,

$$\begin{aligned}
&g^*f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(E,\rho)\cdot\mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\Big)\\
=&(f_g)_*(g_f)^*\Big(\mathrm{ch}_{\mathrm{Ar}}(E,\rho)\cdot\mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\Big)\quad\text{(by A.8)}\\
=&(f_g)_*\Big(\mathrm{ch}_{\mathrm{Ar}}((g_f)^*(E,\rho))\cdot\mathrm{td}_{\mathrm{Ar}}((g_f)^*(T_f,\tau_f))\Big)\quad\text{(by Theorem A.10(i))}\\
=&(f_g)_*\Big(\mathrm{ch}_{\mathrm{Ar}}((g_f)^*(E,\rho))\cdot\mathrm{td}_{\mathrm{Ar}}(T_{f_g},\tau_{f_g}))\Big)\quad\text{(by definition).}
\end{aligned}$$

So by definition and the functorial property for arithmetic characteristic classes, it suffices to show that

$$g^*\Big(f_*E, L^2(\rho,\tau_f)\Big) \simeq \Big((f_g)_* g_f^* E, L^2(g_f^*\rho, \tau_{f_g})\Big).$$

But this is a direct consequence of our f-acyclic condition on E. So Axiom 3 is checked.

(B.6) Now we check Axion 4. This amounts to proving the following

Lemma. *With the same notation as above, let $0 \to E_1 \to E_2 \to E_3 \to 0$ be an exact sequence of f-acyclic vector bundles on X. Then for hermitian metrics ρ_i on E_i, $i = 1,2,3$,*

$$\begin{aligned} &ch'_{\mathrm{BC}}(E_2,\rho_2;f,\tau_f) - ch'_{\mathrm{BC}}(E_1,\rho_1;f,\tau_f) - ch'_{\mathrm{BC}}(E_3,\rho_3;f,\tau_f)\\ =&f_*\Big(ch_{\mathrm{BC}}(E.,\rho.)\cdot td(T_f,\tau_f)\Big) - ch_{\mathrm{BC}}\Big(f_*E., L^2(\rho.,\tau_f)\Big). \end{aligned}$$

Proof. By definition,

$$\begin{aligned} &\mathrm{ch}'_{\mathrm{BC}}(E_2,\rho_2;f,\tau_f) - \mathrm{ch}'_{\mathrm{BC}}(E_1,\rho_1;f,\tau_f) - \mathrm{ch}'_{\mathrm{BC}}(E_3,\rho_3;f,\tau_f)\\ =&f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(E_2,\rho_2)\cdot \mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\Big) - \mathrm{ch}_{\mathrm{Ar}}\Big(f_*E_2, L^2(\rho_2,\tau_f)\Big)\\ &-\Big(f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(E_1,\rho_2)\cdot \mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\Big) - \mathrm{ch}_{\mathrm{Ar}}\Big(f_*E_1, L^2(\rho_1,\tau_f)\Big)\\ &+f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(E_3,\rho_3)\cdot \mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\Big) - \mathrm{ch}_{\mathrm{Ar}}\Big(f_*E_3, L^2(\rho_3,\tau_f)\Big)\Big)\\ =&f_*\Big(\Big(\mathrm{ch}_{\mathrm{Ar}}(E_2,\rho_2) - \mathrm{ch}_{\mathrm{Ar}}(E_1,\rho_1) - \mathrm{ch}_{\mathrm{Ar}}(E_3,\rho_3)\Big)\cdot \mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\Big)\\ &-\Big(\mathrm{ch}_{\mathrm{Ar}}\Big(f_*E_2, L^2(\rho_2,\tau_f)\Big) - \mathrm{ch}_{\mathrm{Ar}}\Big(f_*E_1, L^2(\rho_1,\tau_f)\Big) - \mathrm{ch}_{\mathrm{Ar}}\Big(f_*E_3, L^2(\rho_3,\tau_f)\Big)\Big)\\ =&f_*\Big(a\Big(\mathrm{ch}_{\mathrm{BC}}(E.,\rho.)\Big)\cdot \mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\Big) - a\Big(\mathrm{ch}_{\mathrm{BC}}\Big(f_*E., L^2(\rho.,\tau_f)\Big)\Big)\\ &\qquad\text{(by Theorem 10(iv))}\\ =&f_*\Big(a\Big(\mathrm{ch}_{\mathrm{BC}}(E.,\rho.)\cdot \mathrm{td}(T_f,\tau_f)\Big)\Big) - a\Big(\mathrm{ch}_{\mathrm{BC}}\Big(f_*E., L^2(\rho.,\tau_f)\Big)\Big)\\ &\qquad\text{(by Theorem A.7 and Theorem 10(iii))}\\ =&f_*\Big(\mathrm{ch}_{\mathrm{BC}}(E.,\rho.)\cdot \mathrm{td}(T_f,\tau_f)\Big) - \mathrm{ch}_{\mathrm{BC}}\Big(f_*E., L^2(\rho.,\tau_f)\Big)\\ &\qquad\text{(by Theorem A.7(iii) and (A.8)).} \end{aligned}$$

This completes the proof of the lemma.

(B.7) Similarly, for Axiom 5, we need to show the following

Lemma. *With the same notation as in section 2.E, then*

$$\begin{aligned} &ch'_{\mathrm{BC}}(E,\rho;g\circ f,\tau_{g\circ f}) - g_*\Big(ch'_{\mathrm{BC}}(E,\rho;f,\tau_f)\cdot td(T_g,\tau_g)\Big) - ch'_{\mathrm{BC}}\Big(f_*E, L^2(\rho,\tau_f);g,\tau_g\Big)\\ &= (g\circ f)_*\Big(ch(E,\rho)\cdot td_{\mathrm{BC}}(T.,\tau.)\Big) - ch_{\mathrm{BC}}\Big((g\circ f)_*E; L^2(\rho,\tau_{g\circ f}), L^2(L^2(\rho,\tau_f),\tau_g)\Big). \end{aligned}$$

Proof. By definition,

$$
\begin{aligned}
&\mathrm{ch}'_{\mathrm{BC}}(E,\rho;g\circ f,\tau_{g\circ f})-g_*\Big(\mathrm{ch}'_{\mathrm{BC}}(E,\rho;f,\tau_f)\cdot\mathrm{td}(T_g,\tau_g)\Big)-\mathrm{ch}'_{\mathrm{BC}}\Big(f_*E,L^2(\rho,\tau_f);g,\tau_g\Big)\\
=&(g\circ f)_*\Big(\mathrm{ch}_{\mathrm{Ar}}(E,\rho)\cdot\mathrm{td}_{\mathrm{Ar}}(T_{g\circ f},\tau_{g\circ f})\Big)-\mathrm{ch}_{\mathrm{Ar}}\Big((g\circ f)_*E,L^2(\rho,\tau_{g\circ f})\Big)\\
&-g_*\bigg(\Big(f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(E,\rho)\cdot\mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\Big)-\mathrm{ch}_{\mathrm{Ar}}\Big(f_*E,L^2(\rho,\tau_f)\Big)\Big)\cdot\mathrm{td}(T_g,\tau_g)\bigg)\\
&-\bigg(g_*\Big(\mathrm{ch}_{\mathrm{Ar}}(f_*E,L^2(\rho,\tau_f))\cdot\mathrm{td}_{\mathrm{Ar}}(T_g,\tau_g)\Big)-\mathrm{ch}_{\mathrm{Ar}}\Big(g_*(f_*E),L^2(L^2(\rho,\tau_f),\tau_g))\Big)\bigg)\\
=&(g\circ f)_*\Big(\mathrm{ch}_{\mathrm{Ar}}(E,\rho)\cdot\mathrm{td}_{\mathrm{Ar}}(T_{g\circ f},\tau_{g\circ f})\Big)-\mathrm{ch}_{\mathrm{Ar}}\Big((g\circ f)_*E,L^2(\rho,\tau_{g\circ f})\Big)\\
&-g_*\bigg(\Big(f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(E,\rho)\cdot\mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\Big)-\mathrm{ch}_{\mathrm{Ar}}\Big(f_*E,L^2(\rho,\tau_f)\Big)\Big)\cdot\mathrm{td}_{\mathrm{Ar}}(T_g,\tau_g)\bigg)\\
&-\bigg(g_*\Big(\mathrm{ch}_{\mathrm{Ar}}(f_*E,L^2(\rho,\tau_f))\cdot\mathrm{td}_{\mathrm{Ar}}(T_g,\tau_g)\Big)-\mathrm{ch}_{\mathrm{Ar}}\Big(g_*(f_*E),L^2(L^2(\rho,\tau_f),\tau_g))\Big)\bigg)\\
&\qquad\text{(by Theorem A.7(iii) and Theorem A.10(iii)}\\
&\qquad\text{as } f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(E,\rho)\cdot\mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\Big)-\mathrm{ch}_{\mathrm{Ar}}\Big(f_*E,L^2(\rho,\tau_f)\Big)\in\bar{A}(Y))\\
=&(g\circ f)_*\bigg(\mathrm{ch}_{\mathrm{Ar}}(E,\rho)\cdot\Big(\mathrm{td}_{\mathrm{Ar}}(T_{g\circ f},\tau_{g\circ f})-\mathrm{td}_{\mathrm{Ar}}(T_f,\tau_f)\cdot f^*\Big(\mathrm{td}_{\mathrm{Ar}}(T_g,\tau_g)\Big)\Big)\bigg)\\
&-\bigg(\mathrm{ch}_{\mathrm{Ar}}\Big((g\circ f)_*E,L^2(\rho,\tau_{g\circ f})\Big)-\mathrm{ch}_{\mathrm{Ar}}\Big(g_*(f_*E),L^2(L^2(\rho,\tau_f),\tau_g)\Big)\bigg)\\
&\qquad\text{(by projection formula)}\\
=&(g\circ f)_*\Big(\mathrm{ch}(E,\rho)\cdot\mathrm{td}_{\mathrm{BC}}(T.,\tau.)\Big)-\mathrm{ch}_{\mathrm{BC}}\Big((g\circ f)_*E;L^2(\rho,\tau_{g\circ f}),L^2(L^2(\rho,\tau_f),\tau_g)\Big)\\
&\qquad\text{(by Theorem A.10(iii).}
\end{aligned}
$$

This completes the proof of the lemma.

(B.8) Finally, we are left with checking that $\mathrm{ch}'_{\mathrm{BC}}$ satisfies Axiom 6 for relative Bott-Chern secondary characteristic classes.

Lemma. *With the same notation as in B.2 and as in 3.C.3, we have*

$$
\begin{aligned}
&\lim_{t\to\infty}\Big(\big(ch'_{\mathrm{BC}}(E_t,\rho_t;g_t,\tau_{g_t})-ch'_{\mathrm{BC}}(E_t(-X),\rho'_t;g_t,\tau_{g_t})\big)\\
&+\big(ch_{\mathrm{BC}}((g_t)_*(E_t);L^2(\rho_t,\tau_t),\gamma_t)-ch_{\mathrm{BC}}((g_t)_*(E_t(-X));L^2(\rho'_t,\tau_t),\gamma'_t)\big)\Big)\\
=&\big(ch'_{\mathrm{BC}}(E_\infty,\rho_\infty;g_\infty,\tau_{g_\infty})-ch'_{\mathrm{BC}}(E_\infty(-X),\rho'_\infty;g_\infty,\tau_{g_\infty})\big)+\\
&\big(ch_{\mathrm{BC}}((g_\infty)_*(E_\infty);L^2(\rho_\infty,\tau_\infty),\gamma_\infty)-ch_{\mathrm{BC}}((g_\infty)_*(E_\infty(-X));L^2(\rho'_\infty,\tau_\infty),\gamma'_\infty)\big).
\end{aligned}
$$

Proof. By definition, Theorem A.10(v), and the fact that the L^2-metrics are now replaced by a continuous family of metrics, it is sufficient to show that

$$
\begin{aligned}
&\lim_{t\to\infty}\Big(\mathrm{ch}_{\mathrm{Ar}}(E_t,\rho_t)-\mathrm{ch}_{\mathrm{Ar}}(E_t(-X),\rho'_t)\Big)\cdot\mathrm{td}_{\mathrm{Ar}}(T_t,\tau_{g_t})\\
=&\Big(\mathrm{ch}_{\mathrm{Ar}}(E_\infty,\rho_\infty)-\mathrm{ch}_{\mathrm{Ar}}(E_\infty(-X),\rho'_\infty)\Big)\cdot\mathrm{td}_{\mathrm{Ar}}(T_\infty,\tau_{g_\infty}).
\end{aligned}
$$

But this is a direct consequence of Proposition A.11. In fact, by Proposition A.11, for all $t \in \mathbb{P}^1$,

$$\begin{aligned}&\Big(\mathrm{ch}_{\mathrm{Ar}}(E_t,\rho_t)-\mathrm{ch}_{\mathrm{Ar}}(E_t(-X),\rho_t')\Big)\cdot \mathrm{td}_{\mathrm{Ar}}(T_t,\tau_{g_t})\\ =&i_t^*\bigg(\Big(\mathrm{ch}_{\mathrm{Ar}}(E,D\rho)-\mathrm{ch}_{\mathrm{Ar}}(E,D\rho)\Big)\cdot \mathrm{td}_{\mathrm{Ar}}(T_G(-\mathrm{log}\infty),\tau_G)\bigg).\end{aligned}$$

So, by the fact that there exists a well-defined pull-back morphisms $\mathrm{CH}_{\mathrm{Ar}}(\mathbb{P}^1)\to \mathrm{CH}_{\mathrm{Ar}}(W)$ and that on $\mathbb{P}^1$, the arithmetic cycle $\mathrm{div}_{\mathrm{Ar}}(z):=(0-\infty,-\log|z|^2)$ is rational equivalent to zero, we have

$$\begin{aligned}&\bigg(\Big(\mathrm{ch}_{\mathrm{Ar}}(E(B_XZ),D\rho)-\mathrm{ch}_{\mathrm{Ar}}(E(B_XZ-X\times\mathbb{P}^1),D\rho_0)\Big)\cdot \mathrm{td}_{\mathrm{Ar}}(T_G(-\log\infty),\tau_G)\bigg)\\ &\quad\cdot (W_0-W_\infty,-\mathrm{Log}|z|^2)=0.\end{aligned}$$

Here $\mathrm{Log}|z|$ denotes the pull-back of $\log|z|$ on W. Note that W_∞ has two components which intersects properly, the restriction of $0\to E(B_XZ-X\times\mathbb{P}^1)\to E(B_XZ)$ together with metrics $D\rho'$ and $D\rho$ results an isometry $(E'_\infty,\rho''_\infty)\simeq(E''_\infty,\rho'''_\infty)$, $X\times\mathbb{P}^1$ is away from B_XZ, and $E_\infty-E_\infty(-X)$ is supported on $X\subset\mathbb{P}$,

$$\begin{aligned}&(\Big(\mathrm{ch}_{\mathrm{Ar}}(E(B_XZ),D\rho)-\mathrm{ch}_{\mathrm{Ar}}(E(B_XZ-X\times\mathbb{P}^1),D\rho')\Big)\Big|_{W_\infty}\\ =&(\Big(\mathrm{ch}_{\mathrm{Ar}}(E(B_XZ),D\rho)-\mathrm{ch}_{\mathrm{Ar}}(E(B_XZ-X\times\mathbb{P}^1),D\rho')\Big)\Big|_{\mathbb{P}}.\end{aligned}$$

Hence, in particular,

$$\begin{aligned}&\Big(\mathrm{ch}_{\mathrm{Ar}}(E_\infty,\rho_\infty)-\mathrm{ch}_{\mathrm{Ar}}(E_\infty(-X),\rho'_\infty)\Big)\cdot \mathrm{td}_{\mathrm{Ar}}(T_\infty,\tau_{g_\infty})\\ &-\Big(\mathrm{ch}_{\mathrm{Ar}}(E_0,\rho_0)-\mathrm{ch}_{\mathrm{Ar}}(E_0(-X),\rho'_0)\Big)\cdot \mathrm{td}_{\mathrm{Ar}}(T_0,\tau_{g_0})\\ =&\int_{\mathbb{P}^1}\bigg(\Big(\mathrm{ch}_{\mathrm{Ar}}(E(B_XZ),D\rho)-\mathrm{ch}_{\mathrm{Ar}}(E(B_XZ-X\times\mathbb{P}^1),D\rho')\Big)\\ &\qquad\cdot \mathrm{td}_{\mathrm{Ar}}(T_G(-\log\infty),\tau_G)\bigg)\cdot[\mathrm{Log}|z|^2].\end{aligned}$$

Similarly,

$$\begin{aligned}&\Big(\mathrm{ch}_{\mathrm{Ar}}(E_t,\rho_t)-\mathrm{ch}_{\mathrm{Ar}}(E_t(-X),\rho'_t)\Big)\cdot \mathrm{td}_{\mathrm{Ar}}(T_t,\tau_{g_t})\\ &-\Big(\mathrm{ch}_{\mathrm{Ar}}(E_0,\rho_0)-\mathrm{ch}_{\mathrm{Ar}}(E_0(-X),\rho'_0)\Big)\cdot \mathrm{td}_{\mathrm{Ar}}(T_0,\tau_{g_0})\\ =&\int_{\mathbb{P}^1}\bigg(\Big(\mathrm{ch}_{\mathrm{Ar}}(E(B_XZ),D\rho)-\mathrm{ch}_{\mathrm{Ar}}(E(B_XZ-X\times\mathbb{P}^1),D\rho')\Big)\\ &\qquad\cdot \mathrm{td}_{\mathrm{Ar}}(T_G(-\log\infty),\tau_G)\bigg)\cdot[\mathrm{Log}|\frac{zt}{z-t}|^2].\end{aligned}$$

Easily, one sees that $\lim_{t\to\infty}|\frac{zt}{z-t}|^2=|z|^2$. This completes the proof of the lemma.

(B.9) Recall that $(f : X \to Y; E, \rho; T_f, \tau_f)$ is called a properly metrized datum if $f : X \to Y$ is a smooth morphism of compact Kähler manifolds, (E, ρ) is an f-acyclic hermitian vector bundle, and τ_f is a hermitian metric on the relative tangent bundle T_f of f such that the induced metrics on all fibers of f are Kähler. Then all in all, with the space $\bar{A}$ defined in B.1, what we have already proved is the following

Theorem. (**The Weak Existence and Strong Uniqueness for Relative Bott-Chern Secondary Characteristic Classes**) *There exists a construction ch_{BC} satisfies the six axioms for relative Bott-Chern secondary characteristic classes with values in $\overline{A}$. Moreover, if there are two constructions ch_{BC} and ch'_{BC} which satisfy these six axioms, then, there exists an additive characteristic classes R such that, for all properly metrized data $(f : X \to Y; E, \rho; T_f, \tau_f)$, in $\overline{A}(Y)$,*

$$ch'_{\mathrm{BC}}(E, \rho; f, \tau_f) = ch_{\mathrm{BC}}(E, \rho; f, \tau_f) + a\Big(f_*\Big(ch(E) \cdot td(T_f) \cdot R(T_f)\Big)\Big).$$

(B.10) We end this paper by the following remark. There is in fact another effective construction for relative Bott-Chern secondary characteristic classes by using heat kernels and Mellin transform. Even though such an alternative construction involves only analysis, yet it offers us a strong existence of the relative Bott-Chern secondary characteristic classes, i.e., the relative Bott-Chern classes at the level of $\tilde{A}$. Sure, once this is done, then the arithmetic Grothendieck-Riemann-Roch theorem may be viewed as a direct consequence of the uniqueness for relative Bott-Chern classes. We will study it carefully in another occasion.

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